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AN INVESTIGATION OF SEVERAL FACTORS INVOLVED IN A FINITE DIFFERENCE PROCEDURE FOR ANALYZING THE TRANSONIC FLOW ABOUT HARMONICALLY OSCILLATING AIRFOILS AND WINGS

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16 Abstract <p>Analytical and empirical studies of a finite difference method for the solution of the transonic flow about harmonically oscillating wings and airfoils are presented. The procedure is based on separating the velocity potential into steady and unsteady parts and linearizing the resulting unsteady equations for small disturbances. Since sinusoidal motion is assumed, the unsteady equation is independent of time.</p> <p>Three finite difference investigations are discussed including a new operator for mesh points with supersonic flow, the effects on relaxation solution convergence of adding a viscosity term to the original differential equation, and an alternate and relatively simple downstream boundary condition.</p> <p>A method is developed which uses a finite difference procedure over a limited inner region and an approximate analytical procedure for the remaining outer region.</p> <p>Two investigations concerned with three-dimensional flow are presented. The first is the development of an oblique coordinate system for swept and tapered wings. The second derives the additional terms required to make row relaxation solutions converge when mixed flow is present.</p> <p>Finally, a finite span flutter analysis procedure is described using the two-dimensional unsteady transonic program with a full three-dimensional steady velocity potential.</p> <p>The addition of the viscous term and the revised downstream boundary conditions are the only analyses to have been tested on the computer.</p>					
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1.0 SUMMARY

Several aspects of a finite difference method for solving the unsteady transonic flow about harmonically oscillating wings are analyzed. The procedure is based on separating the velocity potential into steady and unsteady parts and linearizing the resulting unsteady differential equation for small disturbances. The differential equation for the unsteady velocity potential is linear with spatially varying coefficients.

The analyses described herein concern methods of improving the accuracy and efficiency of the finite difference solution. The overstability of the current upwind differencing for supersonic flow is studied for the Klein-Gordon differential equation, which is the equation for the flat plate oscillating in supersonic flow. The operator is shown to be overly stable in that the finite difference solution is attenuated in the downstream direction exponentially in terms of the frequency and the grid size. A stable differencing is derived which has greater accuracy.

The addition of a viscous term has little effect on extending the range of convergence of the relaxation procedure beyond the critical frequency. A simple downstream boundary condition is derived on the assumption that the vortex sheet dominates the flow on the downstream boundary. The results obtained with this boundary condition are indistinguishable from those with the plane wave boundary condition.

Difference equations are derived using an oblique coordinate system which aligns the coordinate lines with the leading and trailing edges of tapered swept wings.

The additional terms required for convergence of row relaxation of three-dimensional mixed flow are also derived.

An approximate method of aeroelastic analysis for high aspect ratio wings using a two-dimensional direct solution with a full three-dimensional steady-state potential is also described.

Except for the addition of the viscous term and the revised downstream boundary conditions, the analyses presented here are yet to be implemented in the program for computing transonic unsteady harmonic flow around airfoils.

2.0 INTRODUCTION

The purpose of the work described in this report is to continue the development of a means for calculating air forces for use in flutter analyses of three-dimensional lifting surfaces in the transonic flight regime. The work concentrates on a particular procedure which assumes small perturbations, the existence of a velocity potential, and simple harmonic motion, and uses finite difference theory to solve the resulting set of partial differential equations. The velocity potential is divided into steady and unsteady parts. The steady potential is calculated using the classic nonlinear small perturbation differential equation. The unsteady potential is then calculated using a linear equation with spatially varying coefficients which depend on the steady flow. This study represents a direct extension of the research described in references 1 through 3. Reference 4 contains the latest results achieved in the investigation while this report covers analyses which for the most part are yet to be implemented. The purpose of these analyses is to improve the efficiency and accuracy of the solution.

Several different finite difference procedures are discussed in section 5.0. Subjects include a new operator for mesh points with supersonic flow which is stable but does not attenuate the initial data, the effects of adding a viscous term to the original differential equation on the convergence of the relaxation solution, and an alternative and relative simple downstream boundary condition.

Section 6.0 presents a method which uses a finite difference procedure over a limited inner region which is matched on the mesh boundary with an approximate linearized solution for the outer region. This has the two-fold purpose of reducing the number of points in the finite difference region and improving the exterior boundary conditions on the mesh. The derivation and a detailed set of equations are included in appendix D.

Section 7.0 discusses two subjects related directly to three-dimensional flow. The first is an oblique coordinate system for swept and tapered wings. The second part discusses additional terms required to make row relaxation solutions converge when mixed flow is present.

Section 8.0 discusses a finite span flutter analysis using the two-dimensional unsteady transonic program with a full three-dimensional steady velocity potential distribution.

3.0 ABBREVIATIONS AND SYMBOLS

a	Streamwise dimension of mesh region; also coordinate of downstream boundary
b	Root semichord of wing or semichord of airfoil; also vertical dimension of mesh region
C_p	Pressure coefficient, $(p - p_o) / (1/2 \rho_o U_o^2)$ where p is the local pressure, p_o the freestream static pressure, and ρ_o the freestream air density
$f(x,y,t)$	Instantaneous wing shape defined by $z_o = \delta f(x,y,t)$
f_o	Undisturbed wing or airfoil shape
f_1	Unsteady contribution to wing or airfoil shape
h	Vertical mesh point spacing
i,j,k I,J,K	x,y,z subscripts and indices for points in the mesh
i	$\sqrt{-1}$
k	Horizontal mesh point spacing
K	Transonic parameter, $(1 - M^2) / (M^2 \epsilon)$
le, LE	Leading edge
M	Freestream Mach number
n,m	Mesh point indices
q	$\omega^2 / \epsilon - i\omega(\gamma - 1)\varphi_{0,xx}$
t	Time in units of b / U_o ; also psuedo time defined by iterations in the complex differential equation for the unsteady potential
TE, te	Trailing edge
U_o	Freestream velocity
x_o, y_o, z_o	Physical coordinates, made dimensionless with the root semichord

x,y,z	Scaled coordinates ($x_0, \mu y_0, \mu z_0$) for the three-dimensional problem; the scaled coordinates for the two-dimensional problem are x and y , with x being the direction of fluid flow
x',y',z'	Variables of integration
x_{le}, x_{te}	Coordinates of leading and trailing edges
\bar{y}	$\sqrt{K}y$
β	$\sqrt{1 - M^2}$
γ	Ratio of specific heats for air
ΔC_p	Jump in pressure coefficient
$\Delta \varphi_1$	Jump in φ_1 at plane of wing or vortex wake
$\Delta \varphi_{1te}$	Jump in φ_1 , at wing trailing edge
δ	Thickness ratio or measure of camber and angle of attack
ϵ	$(\delta / M)^{2/3}$
λ_1	$\omega M / (1 - M^2)$
λ_{1c}	Critical value of λ_1
μ	Scale factor on y_0 and z_0 , $\mu = \delta^{1/3} M^{2/3}$
ξ, η, ζ	Coordinates for swept and tapered wing
$\sigma(x,y)$	Source distribution over mesh boundary for exterior panel method
φ	Complete, scaled perturbation velocity potential; also used for the unsteady potential in finite difference equations. With multiple subscripts, is used as unsteady grid point values of unsteady perturbation potential.
φ_0	Steady scaled perturbation velocity potential
φ_1	Unsteady scaled perturbation velocity potential
ψ	Potential satisfying Klein-Gordon equation. With appropriate subscripts, represents boundary potential for matching inner and outer solution in appendix D.
ω	Angular reduced frequency (semichord times frequency in radians per second divided by the freestream velocity, $\omega b / U$)

4.0 FORMULATION AND SOLUTION

A detailed mathematical derivation of the method for the solution of the unsteady velocity potential for the flow about a harmonically oscillating wing is presented in reference 1. The discussion here will be limited to a brief outline of the procedure for the two-dimensional flow.

The complete nonlinear differential equation was simplified by assuming the flow to be a small perturbation from a uniform stream near the speed of sound. The resulting equation for unsteady flow is

$$\left[K - (\gamma - 1)\varphi_t - (\gamma + 1)\varphi_x \right] \varphi_{xx} + \varphi_{yy} - (2\varphi_{xt} + \varphi_{tt}) / \epsilon = 0 \quad (1)$$

where $K = (1 - M^2) / (M^2 \epsilon)$, M is the freestream Mach number of velocity U_0 in the x -direction, x and y are made dimensionless to the semichord b of the airfoil and the time t to the ratio b/U_0 . With the airfoil shape as a function of time defined by the relation

$$y_0 = \delta f(x, t)$$

the linearized boundary condition becomes

$$\varphi_y = f_x(x, t) + f_t(x, t) \quad (2)$$

The quantity δ is associated with properties of the airfoil (such as maximum thickness ratio, camber, or maximum angle of attack) and is assumed to be small. The coordinate y is scaled to the dimensionless physical coordinate y_0 according to

$$y = \delta^{1/3} M^{2/3} y_0$$

and ϵ is given in terms of δ by

$$\epsilon = (\delta / M)^{2/3}$$

The pressure coefficient is found from the relation

$$C_p = -2\epsilon(\varphi_x + \varphi_t)$$

The preceding differential equation is simplified by assuming harmonic motion and by assuming the velocity potential to be separable into a steady-state potential and a potential representing the unsteady effects. We write for a perturbation velocity potential

$$\varphi = \varphi_0(x, y) + \varphi_1(x, y)e^{i\omega t} \quad (3)$$

and for the body shape

$$y_0 = \delta f(x, t) = \delta \left[f_0(x) + f_1(x)e^{i\omega t} \right]$$

Since the steady-state terms must satisfy the boundary conditions and the differential equation in the absence of oscillations, we obtain

$$\left[K - (\gamma + 1)\varphi_{0x} \right] \varphi_{0xx} + \varphi_{0yy} = 0 \quad (4)$$

with

$$\varphi_{0y} = f_0(x) \quad , \quad y = 0 \quad -1 \leq x \leq 1 \quad (5)$$

On the assumption that the oscillations are small and products of φ_1 may be neglected, equations (1) and (2) with the aid of equations (4) and (5) yield

$$\left\{ \left[K - (\gamma + 1)\varphi_{0x} \right] \varphi_{1x} \right\}_x + \varphi_{1yy} - (2i\omega / \epsilon)\varphi_{1x} + q\varphi_1 = 0 \quad (6)$$

where

$$q = \omega^2 / \epsilon - i\omega(\gamma - 1)\varphi_{0xx}$$

subject to the wing boundary conditions

$$\varphi_{1y} = f_{1x} + i\omega f_1(x) \quad , \quad y = 0 \quad -1 \leq x \leq 1 \quad (7)$$

A computer program for solving the steady-state transonic flow about lifting airfoils based on equations (4) and (5) was developed by Krupp and Murman (refs. 5 and 6). The output of this program or a similar program can be used in computing the coefficients for the differential equation of the unsteady potential. The similarity of the unsteady differential equation to the steady-state equation suggests that the method of column relaxation used by Murman and Krupp for the nonlinear steady-state problem should be an effective way to solve equation (6) for the unsteady potential φ_1 . Note that equation (6) is of mixed type, being elliptic or hyperbolic whenever equation (4) is elliptic or hyperbolic. Central differencing was used at all points for the y derivative and all subsonic or elliptic points for the x derivatives. Backward (or upstream) differences were used for the x derivatives at all hyperbolic points.

The boundary condition that the pressure be continuous across the wake from the trailing edge was found in terms of the jump in potential $\Delta\varphi_1$ to be

$$\Delta\varphi_1 = \Delta\varphi_{1te} e^{-i\omega(x-x_{te})} \quad (8)$$

where $\Delta\varphi_{1te}$ is the jump in the potential at $x = x_{te}$ just downstream of the trailing edge and is determined to satisfy the Kutta condition that the jump in pressure vanish at the trailing edge. The quantity $\Delta\varphi_1$ is also used in the difference formulation for the derivative φ_{1yy} to satisfy continuity of normal flow across the trailing-edge wake.

For the set of difference equations to be determinate, the boundary conditions on the outer edges of the mesh must be specified. In the original unsteady formulation, these boundary conditions were derived from asymptotic integral relations in a manner parallel to that used

by Klunker (ref. 7) for steady flow. A later formulation in reference 3 applies an outgoing plane wave boundary condition to the outer edges of the mesh. This boundary condition is numerically simpler to apply and, on the basis of limited experience, appears to provide equally good correlation.

The preferred numerical approach to solving the resulting large order set of difference equations is a relaxation procedure, which permits the calculation to be made as a sequence of relatively small problems. However, as discussed in preceding NASA reports by the authors (refs. 2 and 3), a significant problem of convergence with the relaxation procedure was encountered which severely limits the range of Mach number and reduced frequency for which solutions may be obtained. The authors currently feel the only practical technique for circumventing these instabilities is a full direct solution where the difference equations are solved "all at once" rather than by line relaxation.

5.0 ALTERNATE FINITE DIFFERENCE PROCEDURES

Three different aspects of the finite difference formulation for two-dimensional unsteady transonic flow are examined in this section. The first concerns an analysis of the finite difference operator currently applied to supersonic mesh points, the second examines the effect on relaxation solution convergence of adding a viscous term to the transonic equation, and the third investigates the application of an alternate downstream boundary condition.

5.1 NINE-POINT OPERATOR FOR SUPERSONIC FLOW

In solutions of difference equations it is important for the operators to lead to stable solutions. Zajac (ref. 8) has shown, however, that the usual upwind differencing for the wave equation leads to an overstable solution in which the solutions decay exponentially with x , the time-like variable. We have extended his results to apply to the Klein-Gordon equation. The details are given in appendix A. For the flat plate in unsteady supersonic flow, the differential equation takes the form of a Klein-Gordon equation

$$\psi_{xx} - \frac{1}{K} \psi_{yy} + \lambda_1^2 \psi = 0 \quad (9)$$

where $K = (M^2 - 1) / (M^2 \epsilon)$, $\lambda_1 = \omega M / (M^2 - 1)$, and $\epsilon = (\delta / M)^{2/3}$.

The function ψ is related to φ_1 by $\varphi_1 = e^{i\lambda_1 M x} \psi$.

Equation (9) is the equivalent to the Helmholtz equation for subsonic flow and is seen to be hyperbolic.

We assume that the region over which the solution is to be found is discretized by a uniform mesh in which the x spacing is k and the y spacing h , with the expression $\psi_{n,m}$ denoting ψ evaluated at $x = nk$ and $y = mh$. Backward differencing on the second derivative with respect to x with central differencing of the y derivative yields for equation (9)

$$\psi_{n,m} - 2\psi_{n-1,m} + \psi_{n-2,m} = p^2 (\psi_{n,m-1} - 2\psi_{n,m} + \psi_{n,m+1}) - k^2 \lambda_1^2 \psi_{n,m} \quad (10)$$

where $p = k / (h \sqrt{K})$. In appendix A, an exact solution of the difference equation (10) is found in the form

$$\psi_{nm} = e^{im\theta} \left(e^{\pm in\tau} \cos^n \tau \right) \quad (11)$$

where $\tau = \tan^{-1} \left[4p^2 \sin^2(\theta/2) + k^2 \lambda_1^2 \right]^{1/2}$. Similarly an exact solution of equation (9) is given by

$$\psi = \exp(i\nu y_m) \cdot \exp\left(\pm i x_n \sqrt{\frac{\nu^2}{K} + \lambda_1^2}\right) \quad (12)$$

Note that this solution is oscillatory without damping.

We compare equation (11) with equation (12) by setting $\theta = h\nu$ in equation (11), expanding

in powers of h and k , and retaining only the first-order terms in h and k . This yields

$$\psi_{nm} = \exp(i\nu y_m) \cdot \exp\left(\pm i x_n \sqrt{\frac{\nu^2}{K} + \lambda_1^2}\right) \cdot \exp\left[-\frac{k}{2}\left(\frac{\nu^2}{K} + \lambda_1^2\right) x_n\right] \quad (13)$$

We see that the difference equation solution has damping in the x direction and because of the terms ν^2 and λ_1^2 the damping is greater for the higher frequency components in the solution and higher reduced frequency in the equation.

A stable difference operator utilizing nine points instead of the usual five will eliminate this excessive damping. We shall use central differencing for the x derivative and for y we use

$$\frac{1}{h^2} \left[a(\psi_{n+1,m+1} - 2\psi_{n+1,m} + \psi_{n+1,m-1}) + (1-2a)(\psi_{n,m+1} - 2\psi_{nm} + \psi_{n,m-1}) \right. \\ \left. + a(\psi_{n-1,m+1} - 2\psi_{n-1,m} + \psi_{n-1,m-1}) \right] \quad (14)$$

where $a > 0$ is the parameter to be determined to make the operator stable. Equation (14) leads to the following difference equation in place of (10):

$$\psi_{nm} - 2\psi_{n-1,m} + \psi_{n-2,m} = p^2 \left[a(\psi_{n+1,m+1} - 2\psi_{n+1,m} + \psi_{n+1,m-1}) \right. \\ \left. + (1-2a)(\psi_{n,m+1} - 2\psi_{n,m} + \psi_{n,m-1}) \right. \\ \left. + a(\psi_{n-1,m+1} - 2\psi_{n-1,m} + \psi_{n-1,m-1}) - k^2 \lambda_1^2 \psi_{nm} \right] \quad (15)$$

We assume a solution to equation (15) in the form

$$\psi_{nm} = \exp(in\theta) \cdot \exp(im\alpha) \quad (16)$$

In appendix B this is found to be stable for

$$a \geq 1/4 \quad (17)$$

$$\text{and } k\lambda_1 \leq 2 \quad (18)$$

Choosing a convenient value of the parameter a subject to equation (17) and k sufficiently small for a given reduced frequency and Mach number will thus ensure a stable operator. Expanding the equation resulting from equation (16) and retaining terms up to first order in h and k yields the solution

$$\psi_{nm} = \exp(i\nu y_m) \cdot \exp\left(i x_n \sqrt{\nu^2 / k + \lambda_1^2}\right) + O(k_1^3, h^3) \quad (19)$$

We see that this is in exact agreement with equation (12). Note also that a influences the solution in the third-order terms in h and k and higher. A simple form of the difference equation results when $a = 1/2$, for then the middle terms in equation (14) vanish. The additional points will not affect the basic diagonal system either for the relaxation solution or the direct solution although computing the matrix is slightly more costly.

5.2 ADDITION OF A VISCOUS TERM

It was suggested that the addition of a viscous term to the differential equation might improve convergence of the relaxation process and extend the frequency limit for which solutions could be obtained. In subsonic regions the viscosity resulting from the first-order truncation terms in the difference equation does play a role in the convergence. Reinforcing this viscosity with an additional term seems like a logical approach to improving convergence. Accordingly, the following differential equation was investigated:

$$\mu \psi_{xxx} + \psi_{xx} + \psi_{yy} + \lambda_1^2 \psi = 0$$

On the upstream boundary the conditions

$$\psi_x = 0 \quad \psi = \sin(\pi y / b)$$

were prescribed with $\psi = 0$ on the other boundaries. The equation was differenced and the program coded along with the amplification factors obtained from a Von Neumann stability analysis. For the coefficient of viscosity μ set to zero, the relaxation converged for values of λ_1 less than the critical values of

$$\lambda_{1c} = \pi \sqrt{\frac{1}{2} + \frac{1}{Kb^2}}$$

where a and b are the horizontal and vertical dimensions of the mesh region, and diverged for λ_1 greater than λ_{1c} as predicted by the Von Neumann analysis. The additional viscosity had very little effect on the convergence even when fairly large values of μ were tried.

5.3 AN ALTERNATE DOWNSTREAM BOUNDARY CONDITION

In the search for simple mesh boundary conditions which would improve the accuracy of the finite difference method for the unsteady subsonic flow over a two-dimensional flat plate, it was reasoned that, far downstream, the flow field is dominated by the vortex wake shed from the wing trailing edge. This boundary condition is easy to formulate since it depends upon the jump in potential at the trailing edge required to satisfy the Kutta condition. Thus each difference equation for the column of grid points next to the downstream boundary would contain four additional terms involving the four values of the potential in the neighborhood of the trailing edge upon which the jump in potential depends. The potential resulting from the wake is an infinite integral of a Hankel function and for two-dimensional flow is given by equation (109) in reference 1 in the form

$$\varphi_1(x, y_1) = \frac{\Delta \varphi_{te}}{4i} \int_1^\infty \exp[-i\omega(x' - 1)] \cdot \psi_{y_1}' \Big|_{y_1' = 0} dx'$$

where $\psi = \exp[i\lambda_1 M(x - x')] \cdot H_0^{(2)} \left[\lambda_1 \sqrt{(x - x')^2 + (y_1 - y_1')^2} \right]$, $y_1 = \sqrt{K}y$

and $H_0^{(2)}$ is the Hankel function of the second kind. If instead of φ_1 the pressure function

$$\varphi_{1x} + i\omega \varphi_1$$

is prescribed on the downstream boundary, then the resulting integral in the equation obtained from applying the operator $\frac{\partial}{\partial x} + i\omega$ can be integrated in closed form. From equation (C-4) in appendix C this is seen to be

$$\varphi_{1x} + i\omega\varphi_1 = \frac{-i\Delta\varphi_{te}}{4} \exp[i\lambda_1 M(x - x_{te})] \cdot \lambda_1 \sqrt{K} y H_1^{(2)}(\lambda_1 r) / r$$

where $r = \sqrt{(x - x_{te})^2 + Ky^2}$ and x_{te} is the x coordinate of the trailing edge.

The coefficients in the difference equations on the column adjacent to the downstream boundary for the potentials

$$\varphi_{i_1-1, j_m-1}, \varphi_{i_1-1, j_m}, \varphi_{i_1, j_m-1}, \varphi_{i_1, j_m}$$

where j_m is the y grid index in the row adjacent to the wing and wake and i_1 is the x grid index for the point at the trailing edge, are developed in appendix C. The equations were derived assuming that φ_1 is antisymmetric about the line $y = 0$, corresponding to the wing and wake, for the purpose of testing the concept as economically as possible. The resulting pressures on the wing differed insignificantly from the results obtained by assuming an outgoing plane wave boundary condition on the downstream boundary. Furthermore, the pressures are not sensitive to the location of the downstream boundary. It therefore appears that the outgoing plane wave boundary conditions produce little reflection back to the airfoil although the distance from the wing to the downstream boundary is not so great that one should expect the disturbance to resemble very closely a true plane wave.

6.0 A FAR-FIELD MATCHING METHOD FOR TRANSONIC UNSTEADY FLOW USING THE DIRECT SOLUTION

Chen, Dickson, and Rubbert (ref. 9) developed a method for matching the far-field boundary of the transonic steady finite difference mesh with an analytic outer solution. Their method has the advantage of imposing analytic boundary conditions at infinity while permitting a considerable reduction in the size of the mesh region. The mesh need extend outward only to where the flow is subsonic and linearized theory is valid, rather than to a distance at which the approximate evaluations of the outgoing wave boundary conditions, or alternatively the Klunker-type boundary conditions, are valid. The solution in reference 9 was obtained using relaxation procedures. However, it is possible to obtain the far-field matching solution and the inner finite difference solution in a single step.

Reducing the size of the solution will facilitate refining the mesh size and this is necessary for obtaining suitable accuracy with the finite difference method at higher frequencies. Also, the reduction in mesh points may be enough to make the direct solutions practical for three-dimensional problems. Alternatively, this matching procedure may provide better boundary conditions at the higher reduced frequencies, although the need for this is somewhat reduced by the improved results presented in reference 4.

The procedure is applicable to both two- and three-dimensional flow although the following derivation is for the two-dimensional problem only. In this section the basic ideas are sketched briefly but a detailed derivation is presented in appendix D. In section D.1, the basic integrals are discussed. In section D.2, the basic functions for the panel source distribution are presented and the form of the influence coefficient integrals defined. Far- and near-field approximations of the integrals are analyzed in sections D.3 and D.4. Subsequent sections formulate the boundary conditions, the wake, and the matrix coefficients in sufficient detail for coding into the direct two-dimensional solution.

The derivation of the matrix elements are for a doubly symmetric grid distribution and symmetric steady flow so that the method may be evaluated as economically as possible. The basic integrals are simplified to the extent of requiring a single coded subroutine. In section D.11, the formulas for those matrix coefficients required by the outer solution are defined in a simple form suitable for coding.

Following reference 9, an acoustic source distribution is prescribed on the outer edge of the mesh and a single vortex line imposed on the wake. The source strength is determined to satisfy continuity between inner and outer solutions of the normal component of the velocity and the velocity potential at the outer mesh boundary. The vortex line accounts for the jump in potential of the wake. We assume that the velocity potential of the outer solution satisfies the linearized differential equation for the harmonic unsteady flow of a gas given by

$$K\varphi_{1_{xx}} - 2i(\omega / \epsilon)\varphi_{1_x} + \varphi_{1_{yy}} + (\omega^2 / \epsilon)\varphi_1 = 0$$

From equation (109) of Ehlers (ref. 1), the solution to this equation given by a source

distribution on the boundary of the mesh and a doublet sheet from the wing trailing edge takes the form

$$\varphi_1 = -\frac{e^{i\lambda_1 Mx}}{4i} \int_{-a}^a [\sigma_u \psi_u - \sigma_d \psi_d] dx' - \frac{e^{i\lambda_1 Mx}}{4i} \int_{-b}^b [\sigma_\ell \psi_\ell - \sigma_r \psi_r] dy' + \Delta\varphi_{1te} \cdot \chi(x,y) \quad (20)$$

where $\psi_u = H_0^{(2)} \left[\lambda_1 \sqrt{(x-x')^2 + K(y-b)^2} \right]$ $\psi_\ell = H_0^{(2)} \left[\lambda_1 \sqrt{(x-a_1)^2 + K(y-y')^2} \right]$ with similar expressions for ψ_d and ψ_r . Here χ is the potential induced by the trailing vortex sheet

$$\chi = -\frac{1}{4i} \int_1^\infty e^{-i\omega(x'-1)} \psi_y dx'$$

$$\text{with } \psi = H_0^{(2)} \left(\lambda_1 \sqrt{(x-x')^2 + Ky^2} \right)$$

The quantity σ is the source strength, a and b are the width and height of the mesh shown by the heavy line in figure 1, u and d denote upper and lower, and ℓ and r , left and right boundaries, respectively. For each $1 < i < i_{\max}$ designating the column for the upper and lower boundaries, and each $1 < j < j_{\max}$ on the side boundaries, we match boundary conditions on the normal derivative and on the potential at each boundary point with the finite difference solution. The number of values for the potential φ_1 from the finite difference equations is $i_{\max} j_{\max} - 4$.

At each outer boundary point we assign a value of the source and construct a piecewise linear distribution of the source strength on the mesh boundary, utilizing for each boundary point the elementary singularity distribution in figure 2. The velocity potential in equation (20) after the integration takes the form

$$\varphi = \sum_{n=2}^{i_{\max}-1} [\sigma_{un} \varphi_{un} - \sigma_{dn} \varphi_{dn}] + \sum_{n=2}^{j_{\max}-1} [\sigma_{\ell n} \varphi_{\ell n} - \sigma_{rn} \varphi_{rn}] + \Delta\varphi_{1te} \cdot \chi(x,y) \quad (21)$$

where the φ_n terms are the functions of x and y resulting from integrating the basis function of figure 2 over the range x_{n-1} to x_{n+1} or y_{n-1} to y_{n+1} . The jump in the potential $\Delta\varphi_{te}$ is given by a linear combination of values of the potential φ_1 at points in the neighborhood of the trailing edge.

We now match the solution of equation (20) with the inner finite difference solution. On the upper boundary we write for the velocity potential

$$\varphi = (\varphi_{ij_{\max}} + \varphi_{i,j_{\max}-1}) / 2 = F_i^{(u)}(\sigma, \Delta\varphi_{1te}), \quad i = 2, 3, \dots, i_{\max}-1$$

where $F_i^{(u)}(\sigma, \Delta\varphi_{te})$ is a linear function of the σ 's and the φ 's by equation (21) evaluated at the boundary point $x = x_i, y = b$ on the upper boundary. In the same manner, for the lower boundary, we obtain

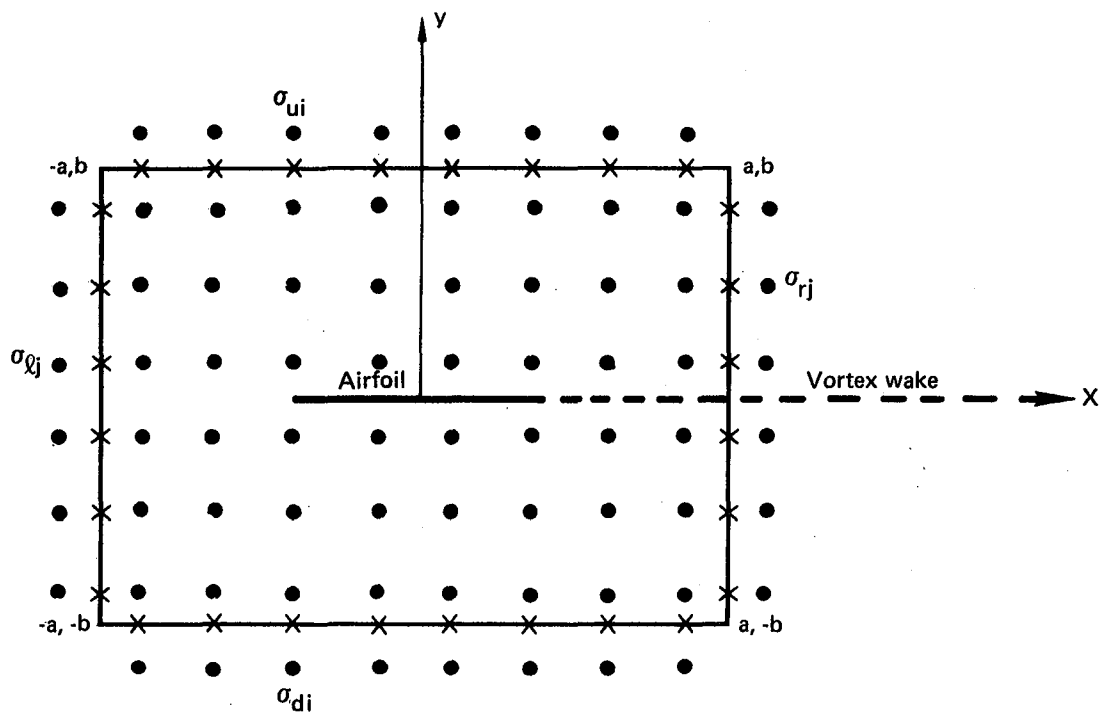


Figure 1.—Inner and Outer Solution Boundary

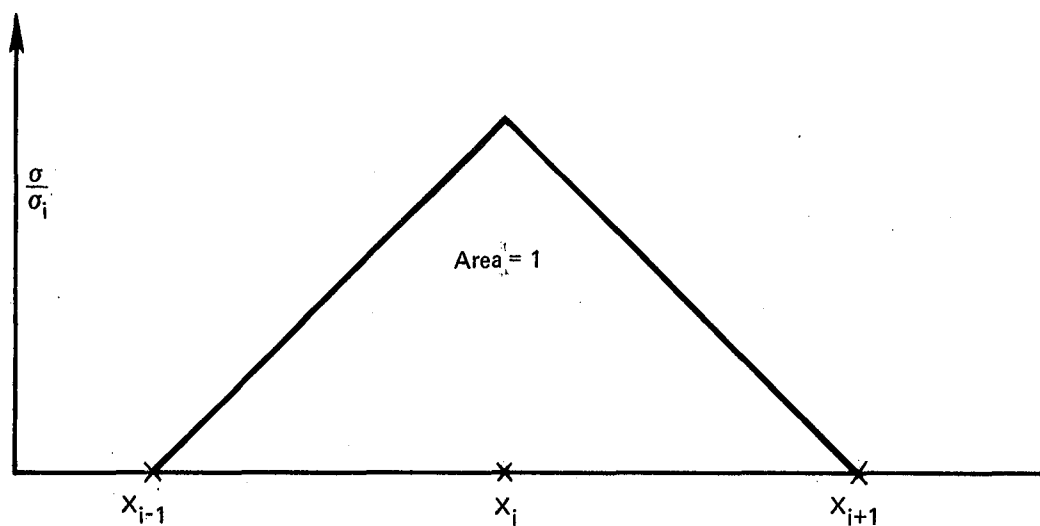


Figure 2.—Elementary Singularity Strength Distribution for Two-Dimensional Linear Varying Sources on Matching Boundary

$$(\varphi_{i1} + \varphi_{i2}) / 2 = F_i^{(d)}(\sigma, \Delta\varphi_{1_{te}}), \quad i = 2, 3, \dots, i_{\max}-1$$

For the left boundary

$$(\varphi_{1j} + \varphi_{2j}) / 2 = F_j^{(\ell)}(\sigma, \Delta\varphi_{1_{te}}), \quad j = 2, 3, \dots, j_{\max}-1$$

For the right boundary

$$(\varphi_{i_{\max}j} + \varphi_{i_{\max}-1,j}) / 2 = F_j^{(r)}(\sigma, \Delta\varphi_{1_{te}}), \quad j = 2, 3, \dots, j_{\max}-1$$

Similarly, we evaluate the normal derivative on each of the mesh boundaries and obtain:

For the upper boundary

$$\varphi_y = \frac{\varphi_{ij_{\max}} - \varphi_{i,j_{\max}-1}}{y_{j_{\max}} - y_{j_{\max}-1}} = G_i^{(u)}(\sigma, \Delta\varphi_{1_{te}}), \quad i = 2, 3, \dots, i_{\max}-1$$

For the lower boundary

$$\varphi_y = \frac{\varphi_{i2} - \varphi_{i1}}{y_2 - y_1} = G_i^{(d)}(\sigma, \Delta\varphi_{1_{te}}), \quad i = 2, 3, \dots, i_{\max}-1$$

For the left side

$$\varphi_x = \frac{\varphi_{2j} - \varphi_{1j}}{x_2 - x_1} = G_j^{(\ell)}(\sigma, \Delta\varphi_{1_{te}}), \quad j = 2, 3, \dots, j_{\max}-1$$

For the right side

$$\varphi_x = \frac{\varphi_{i_{\max}j} - \varphi_{i_{\max}-1,j}}{x_{i_{\max}} - x_{i_{\max}-1}} = G_j^{(r)}(\sigma, \Delta\varphi_{1_{te}}), \quad j = 2, 3, \dots, j_{\max}-1$$

It is easily seen that the preceding systems of equations, along with the finite difference equations, yields

$$i_{\max}j_{\max} + 2(i_{\max} + j_{\max}) - 12$$

equations for the same number of variables to be determined. The increase in the number of variables required by the matching procedure is

$$2(i_{\max} + j_{\max}) - 12$$

and is offset by the considerable reduction in the size of the mesh region. Because of the wavelike nature of the solution for the unsteady flow, using large mesh sizes near the outer boundaries of the finite difference mesh has been found to lead to poor representation of the flow field, resulting in inaccurate pressures on the wing. By decreasing the size of the mesh region, finer grids are possible with the same number of mesh points. The coefficients of the additional terms in the system of equations are derived in considerable detail in appendix D.

It seems worthwhile to make some assessment of the computer resources required to apply the inner and outer matching. In appendix D, the number of integrals was reduced by assuming a grid which has two lines of symmetry. For a grid with i_{\max} x points and j_{\max} y points, the number of integrals N_I to be evaluated for the symmetrical problem is

$$N_I = [i_{\max} + 2(j_{\max} - 1) - 2]^2$$

Many of these integrals need not be calculated by actually performing the integration using the Bessel function routines. In regions where the grid is fine in the far field, the integrals for every second or third grid point need be computed with the intermediate points evaluated by interpolation. The integrals are more complicated than the coefficients in the potential solution and hence must be calculated more efficiently. Chen et al. (ref. 9) obtained a considerable reduction in computing cost as well as improvement in accuracy. Much of the reduction in cost will come from the smaller mesh region made possible by the better mesh boundary conditions. For the higher frequencies where the grid spacing must be fine even in the outer field, this smaller mesh region should result in a considerably smaller matrix equation to be solved. Unfortunately, the equations involving the values of the source at the boundary grid points contain nonzero coefficients for most of the source values. Hence, the use of the inner and outer matching procedure introduces to the matrix of coefficients a vertical strip of nonzero elements which nearly eliminates the banded property of the original finite difference matrix.

7.0 INVESTIGATIONS FOR THE THREE-DIMENSIONAL PROBLEM

Three-dimensional investigations for this report were limited to two problems. The first concerned developing a coordinate transformation for swept wings that concentrated grid points in regions of large gradients of φ_1 . The results of this study, which are based on transformations used for steady state, are presented in section 7.1, and detailed derivations are presented in appendix E. Previous studies have resulted in a derivation of a coordinate transformation for swept but untapered wings (ref. 1) and development of a three-dimensional program using a cartesian coordinate grid (ref. 2). In reference 2 it was also shown that for the two-dimensional problem, row relaxation converged more rapidly than column relaxation but that additional terms were required for points at which the equation was hyperbolic in order for the relaxation to converge. These terms for the three-dimensional problem are discussed in section 7.2, with a detailed derivation presented in appendix F.

7.1 AN OBLIQUE COORDINATE SYSTEM FOR SWEEPED AND TAPERED WINGS

The three-dimensional unsteady transonic flow program described in reference 2 utilizes a rectangular grid. Better accuracy with fewer grid points can be achieved by using an oblique coordinate system chosen to align the leading and trailing edges with coordinate lines and hence provide the capability of finer grid spacing along these edges. The transformation will also make the unsteady program more compatible with the steady program.

In the same manner as Bailey and Ballhaus (ref. 10) we consider a transformation of the form

$$\xi = \frac{x - x_{le}(y)}{c(y)} = g(x,y) \quad (22)$$

$$\eta = y$$

$$\zeta = z$$

where $c(y)$ is the chord of the wing at the station y and $x_{le}(y)$ is the leading edge of the wing planform. Thus $\xi = 0$ is the coordinate representing the wing leading edge while $\xi = 1$ is the trailing edge.

The coordinates ξ, η must be defined beyond the wing tip. To achieve this, the wing leading edge is extended all the way to the mesh boundary by a straight line having the same slope as the wing leading edge at the tip. To ensure that ξ, η is single valued in the region beyond the wing tip, the trailing edge is continued analytically beyond the tip by a quadratic whose slope varies continuously from the trailing edge value at the tip to a straight line parallel to the leading edge extension, as shown in figure 3. Thus the functions $x_{le}(y)$ and $c(y)$ and their derivatives are defined over the entire η range of the mesh region.

Under the transformation of equation (22), the transonic unsteady differential equation was obtained in appendix E, and in conservation form can be expressed as

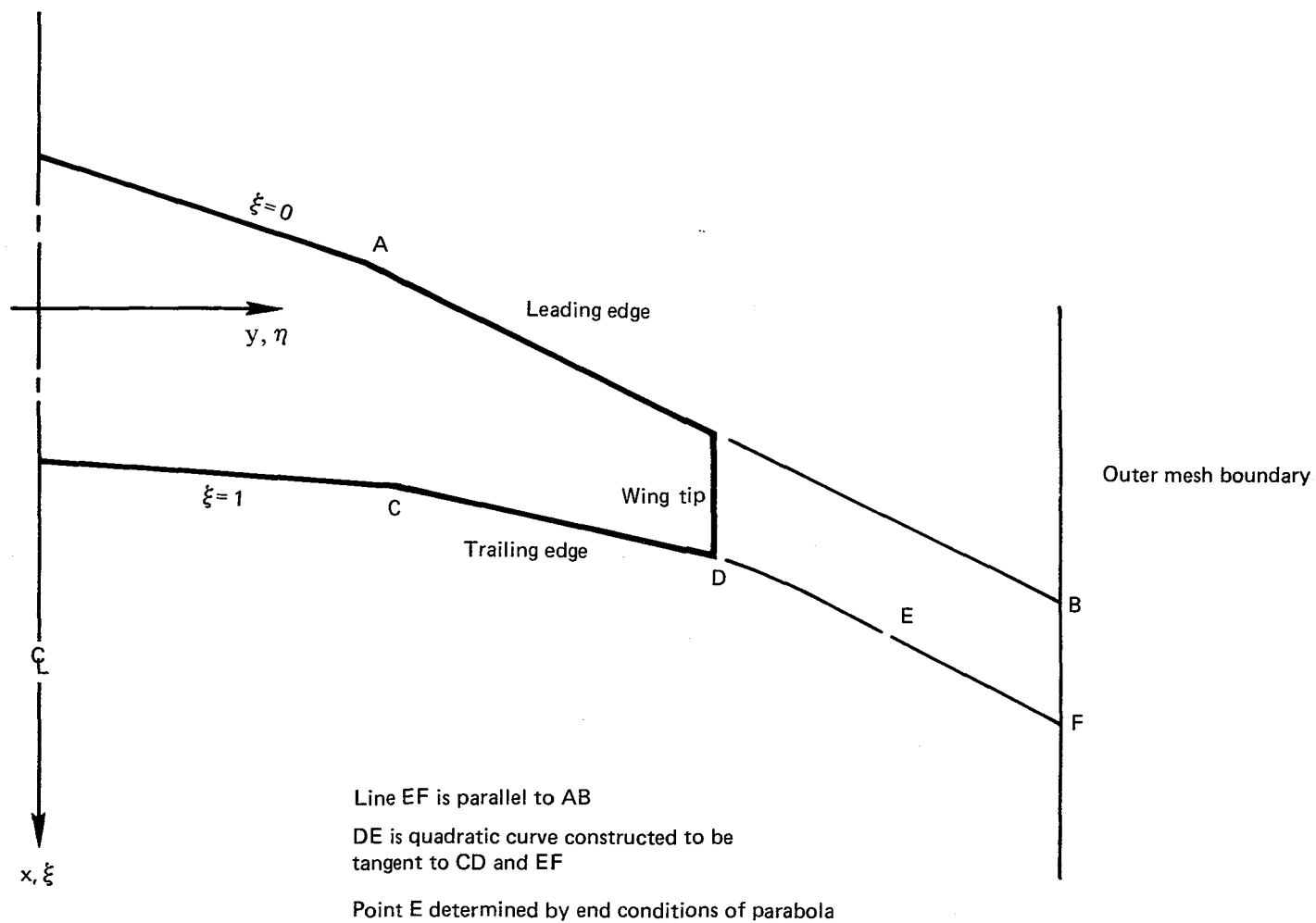


Figure 3.—Oblique Coordinate System for Swept and Tapered Wings

$$\frac{\partial}{\partial \xi} \left[(\mu^2 u + \nu^2) \varphi_{1\xi} + (\nu - 2i\omega / \epsilon) \varphi_{1\eta} + \nu \varphi_{1\eta} \right] + \frac{\partial}{\partial \eta} \left[(1 + c' / c) \varphi_{1\eta} + \nu \varphi_{1\xi} \right] \quad (23)$$

$$+ \varphi_{1\xi\xi} + \left[q + c' / c - (c' / c)' \right] \varphi_1 = 0$$

where $\mu = 1 / c(\eta)$ and $\nu = -\xi c' / c - x'(\eta) / c(\eta)$. A simpler nonconservation form is given by

$$\mu \frac{\partial}{\partial \xi} \left[\mu u \varphi_{1\xi} - 2i\omega \varphi_{1\eta} / \epsilon \right] + \nu \frac{\partial}{\partial \xi} (\varphi_{1\eta} + \nu \varphi_{1\xi}) + \frac{\partial}{\partial \eta} (\varphi_{1\eta} + \nu \varphi_{1\xi}) + \varphi_{1\xi\xi} + q \varphi_1 = 0 \quad (24)$$

The conditions that the equation be hyperbolic for both forms is

$$\mu^2 u + \nu^2 < 0 \quad (25)$$

This is the condition that Bailey and Ballhaus used at first to determine when to employ upstream differencing in the derivatives. They found that upwind differencing for all supersonic points was required to capture the shock.

On the wing root plane we must apply the boundary condition of symmetry $\varphi_{1y} = 0$. In terms of the oblique coordinates this condition becomes

$$\varphi_{1\eta} + \nu \varphi_{1\xi} = 0 \quad (26)$$

This boundary condition is applied to the difference equation for points along the wing root and leads to some simplification. The boundary conditions on the wing and on the wake are unchanged under the transformation.

Equations (23) and (24) may be differenced in the same way as described in reference 1 and formulas are presented in the appendix E. Because of the cross-derivative terms, the grid point pattern used to represent the difference operator contains the eleven points shown in figure 4 instead of the seven points for the operator in cartesian coordinates in figure 5.

7.2 ROW RELAXATION FOR THREE-DIMENSIONAL FLOW

In reference 2 it was found that row relaxation for the two-dimensional solution of the unsteady velocity potential converges more rapidly than column relaxation. When the flow is completely subsonic the same difference equations may be used for either row or column relaxation. However, for mixed flows, the row relaxation will diverge unless additional time-like terms are added at supersonic grid points.

Following Jameson in references 11 and 12, we introduce the time-like variable associated with the iteration process in the form of

$$\varphi_{ijk}^{(n)} - \varphi_{ijk}^{(n-1)} = \Delta t \cdot \left(\varphi_{ijk}^{(n)} \right)_t \quad (27)$$

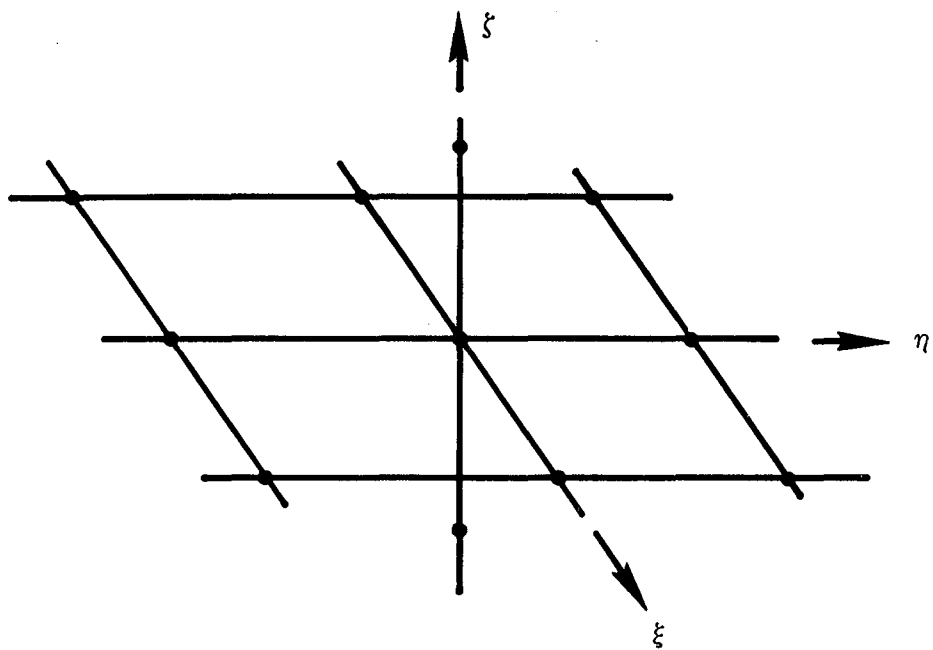


Figure 4.—Eleven-Point Difference Operator for Swept Wings

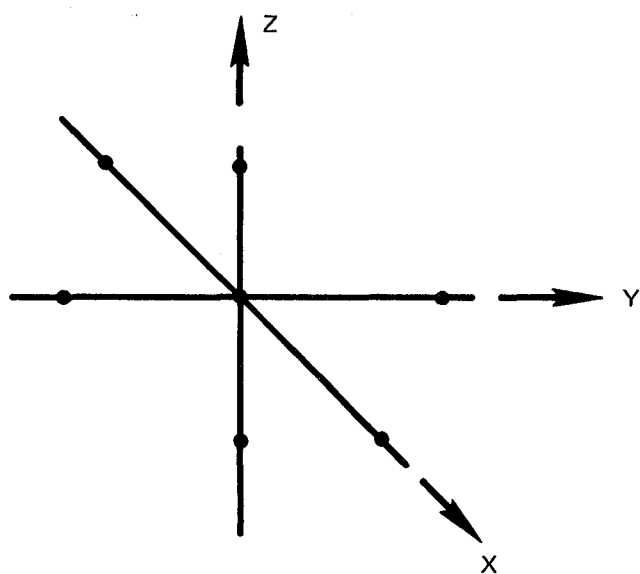


Figure 5.—Seven-Point Difference Operator for Cartesian Coordinates

and obtain the following differential equation by taking the limit as $\Delta t, \Delta x, \Delta y, \Delta z$ go to zero in the difference equation about the point ijk (see eq. (F-8) in app. F).

$$\left[u\varphi_{1x} - 2i(\omega / \epsilon)\varphi_1 \right]_x + \varphi_{1yy} + \varphi_{1zz} - 2 \frac{\Delta t}{\Delta y_j} \left(\varphi_{1yt} + \frac{\Delta y_j}{\Delta z_k} \varphi_{1zt} + \beta_3 \varphi_{1t} \right) + q\varphi_1 = 0 \quad (28)$$

By a transformation of t in the form

$$\tau = \alpha x + \beta y + \gamma z + t \quad (29)$$

equation (28) can be converted to

$$\left[u\varphi_{1x} - 2i(\omega / \epsilon)\varphi_1 \right] + \varphi_{1yy} + \varphi_{1zz} - a_{jk}\varphi_{\tau\tau} + b_{jk}\varphi_{1\tau} + q\varphi_1 = 0 \quad (30)$$

where $a_{jk} > 0$. Since $u > 0$ at supersonic points, the resulting differential equation is not strictly hyperbolic in τ as it is for subsonic point.

The terms $\varphi_{1\tau}$ and $\varphi_{1\tau\tau}$ are truncation terms resulting from differencing the x, y, z derivatives in the conventional manner. To render the φ_{1xx} term time-like we must add φ_{1xt} differences to change the sign of $\varphi_{1\tau\tau}$ and a φ_{1t} difference to cancel the $\varphi_{1\tau}$ term in equation (30). The derivation is presented in detail in appendix F and the equation to be differenced is (eq. (F-19) of app. F)

$$\left[u\varphi_{1x} - 2i(\omega / \epsilon)\varphi_1 \right]_x + \varphi_{1yy} + \varphi_{1zz} - 2 \frac{\Delta t}{\Delta y_j} \left[\beta_1 \varphi_{1xt} + \beta_2 \varphi_{1t} \right] + q\varphi_1 = 0 \quad (31)$$

where

$$\beta_1 = c \sqrt{-u} \sqrt{1 + \beta_{jk}^2}, \quad c > 1 \quad (32)$$

$$\beta_2 = \left[\beta_1 (u_x - 2i \omega / \epsilon) / 2u \right] - \beta_3$$

$$\beta_3 = -\Delta y_j \left[b_j + b_k - (a_j + a_k) (r - 1) \right] / r$$

$$\Delta y_j = y_{j+1} - y_{j-1}$$

$$\beta_{jk} = \Delta y_j / (z_{k+1} - z_{k-1})$$

r = the relaxation factor

The factors in equation (32) for two-dimensional flow are obtained from equation (32) by dropping all terms with subscript k .

Row relaxation has the same frequency limitation as column relaxation but its greater efficiency may make it worthwhile for frequency ranges in which it converges, while going to some form of direct solution for the higher frequencies.

8.0 FORMULATION OF AEROELASTIC ANALYSIS

References 1 to 4 describe a practical procedure for calculating transonic air forces for harmonically oscillating airfoils. The frequency limitation problem discussed in references 2 and 3 appears to have been overcome to the point where combinations of Mach number and reduced frequency of practical interest in flutter can be handled. The size capability of the pilot two-dimensional program has been increased to work with a practical number of mesh points for these analyses. Also, the solution program has been modified to treat multiple right-hand sides efficiently. However, due to the large size of the matrix inverse which is required, this procedure does not, as yet, appear to be practical for full three-dimensional configurations. The full three-dimensional problem involves an inverse of a $50\,000^{\text{th}}$ to $100\,000^{\text{th}}$ order complex matrix. This may eventually be practical through the use of the new vector machines, or through use of sparse matrix concepts. The following paragraphs discuss one use of the harmonic finite difference procedure in flutter analyses.

It should be emphasized that the problem formulation provides superposable pressure distributions which can be used directly in conventional (e.g., V-g) flutter analyses. The flutter equations in matrix form and applicable to both two- and three-dimensional flows are presented in detail in section 10.0 of reference 3.

Use of the direct solution program of this report for practical two-dimensional flutter problems appears to be feasible. It is, of course, highly desirable to extend the harmonic analysis to full three-dimensional flow. However, a reasonable alternative may be to use the two-dimensional program to calculate the unsteady pressures at several spanwise stations with the equation coefficients being determined from the three-dimensional steady-state velocity potential. This would make use of the current capability and include the major three-dimensional effects of the shock and boundary layer through the steady-state potential, and could prove in the long run to be a valid economical alternative to the full three-dimensional calculation which would be much more expensive in terms of computer resources. The procedure may be summarized in the following steps:

1. Calculate the steady three-dimensional velocity potential distribution using a standard small perturbation program such as that of Ballhaus and Bailey.
2. Use the two-dimensional unsteady program with the three-dimensional steady-state potential to calculate sectional harmonic pressure distributions at a set of spanwise stations. Using the steady potential ensures that the three-dimensional shock effects are incorporated in the results.
3. Form a three-dimensional pressure distribution from the two-dimensional section distributions. Additional finite span corrections could be introduced at this time. These corrections could be based, for example, on empirical data or steady-state analytical data.
4. Calculate generalized air forces and perform flutter analysis.

APPENDIX A

OVERSTABILITY OF THE CANONICAL UPWIND SUPERSONIC OPERATOR APPLIED TO THE KLEIN-GORDON EQUATION

A.1 INTRODUCTION

When partial differential equations are solved by numerical methods, an area of particular concern is the stability of the numerical operators employed. In the case of hyperbolic equations, in particular, it is required that the operators be stepwise stable, i.e., that errors at one stage are not magnified as the solution is stepped along in time (or in a time-like direction). Such stability may ordinarily be established by a Von Neumann analysis.

It has been observed by Zajac in reference 8 that some operators may be so stable that the correct numerical solution is distorted by being attenuated in stepping along. He has called this phenomenon overstability. The situation here is that while the numerical solution will converge to the true solution as the step size is refined, for a given step size the error may compare poorly with that obtained using a less strongly stable operator.

A.2 ANALYSIS OF THE NUMERICAL SOLUTION OF THE KLEIN-GORDON EQUATION USING THE SUPERSONIC OPERATOR

A.2.1 DEFINITION

The Klein-Gordon equation

$$\psi_{xx} - \frac{1}{K} \psi_{yy} + \lambda_1^2 \psi = 0 \quad (\text{A-1})$$

where

$$K = (M^2 - 1) / (M^2 \epsilon), \quad \lambda_1 = \omega M / (M^2 - 1)$$

and

$$\epsilon = (\delta / M)^{2/3}$$

bears the same relation to the flat plate equation for supersonic flow as the Helmholtz equation does for subsonic flow. Observe that when $\lambda_1 = 0$, the K-G equation becomes the wave equation as, analogously, the Helmholtz equation becomes Laplace's equation. In the supersonic case, however, x and y are not treated identically in the discretization, but rather the time-like character of x is considered and a backward difference operator is used.

A.2.2 DISCRETIZATION

We suppose the region over which equation (A-1) is to be solved to be discretized by a mesh such that k is the spacing in the x direction and h is the spacing in the y direction. With the mesh point which is the n^{th} in the x direction and m^{th} in the y direction, there is associated a value ψ_{nm} which is an approximation to $\psi(nk, mh)$, i.e., to the solution at this mesh point.

Using the backward and central difference operators in the x and y directions, respectively, equation (A-1) becomes

$$\left(\psi_{nm} - 2\psi_{n-1,m} + \psi_{n-2,m}\right) / k^2 = \frac{1}{K} \left(\psi_{n,m-1} - 2\psi_{nm} + \psi_{n,m+1}\right) / h^2 - \lambda_1^2 \psi_{nm}$$

or,

$$\psi_{nm} - 2\psi_{n-1,m} + \psi_{n-2,m} = p^2 \left(\psi_{n,m-1} - 2\psi_{nm} + \psi_{n,m+1}\right) - k^2 \lambda_1^2 \psi_{nm} \quad (A-2)$$

where

$$p = k / (h\sqrt{K})$$

A.2.3 STABILITY ANALYSIS

The exact solution for the difference equation (A-2) is

$$\psi_{nm} = \alpha^n e^{im\theta} \quad (A-3)$$

Substitution of (A-3) into (A-2) yields

$$e^{im\theta} \left(\alpha^n - 2\alpha^{n-1} + \alpha^{n-2}\right) = p^2 \alpha^n \left[e^{i(m-1)\theta} - 2e^{im\theta} + e^{i(m+1)\theta}\right] - k^2 \lambda_1^2 \alpha^n e^{im\theta}$$

From which on division by

$$\alpha^{n-2} e^{im\theta}$$

we have

$$\left(\alpha^2 - 2\alpha + 1\right) = p^2 \alpha^2 \left[-4 \sin^2(\theta/2) - k^2 \lambda_1^2\right]$$

or

$$\left[1 + 4p^2 \sin^2(\theta/2) + k^2 \lambda_1^2\right] \alpha^2 - 2\alpha + 1 = 0 \quad (A-4)$$

which is a quadratic equation in α . Since the coefficient of α^2 is always ≥ 1 , we can define

$$\cos \tau = \left[1 + 4p^2 \sin^2(\theta/2) + k^2 \lambda_1^2\right]^{-1/2} \quad (A-5)$$

for $\theta \leq \tau < \pi/2$. Then (A-4) may be written as

$$\alpha^2 - 2(\cos \tau) \alpha + \cos^2 \tau = 0 \quad (A-6)$$

Solving for α we have that

$$\alpha = e^{\pm i\tau} \cos \tau \quad (A-7)$$

Thus since $|\alpha| \leq 1$ for all τ , the operator given in equation (A-2) is unconditionally step-wise stable.

A.2.4 OVERSTABILITY ANALYSIS

In this section we show that the difference scheme used in obtaining equation (A-2) from

equation (A-1) yields an overstable operator.

First, from equations (A-3), (A-5), and (A-7) we observe that exact solutions of the difference equation (A-1) are given at mesh point (n,m) by

$$\psi_{nm} = \alpha^n e^{im\theta}$$

or

$$\psi_{nm} = e^{im\theta} \left(e^{\pm in\tau} \cos \frac{n}{\tau} \right) \quad (\text{A-8})$$

for any real θ , where

$$\tau = \tan^{-1} \left[4p^2 \sin^2 \left(\frac{\theta}{2} \right) + k^2 \lambda_1^2 \right]^{\frac{1}{2}} \quad (\text{A-9})$$

Second, we observe that exact solutions of the differential equation (A-1) are given by

$$\psi_{nm}^T = \exp(i\nu y_m) \cdot \exp \left(\pm ix_n \sqrt{\frac{2\nu^2}{K} + \lambda_1^2} \right) \quad (\text{A-10})$$

for any real ν . Observe that the solutions in equation (A-10) oscillate without damping.

Let us try to compare equations (A-8) and (A-10). This is facilitated by letting $\theta = h\nu$ in (A-8). For the first factor we have then

$$e^{im\theta} = e^{imh\nu} = \exp(i\nu y_m)$$

with $y_m = mh$. This is the same as the first factor of (A-10).

Next consider the second factor of equation (A-8), $e^{\pm in\tau}$. Letting $\theta = h\nu$ in equation (A-9), we have

$$\tau = \tan^{-1} \left[4 \frac{k^2}{h^2 K} \sin^2 \left(\frac{h\nu}{2} \right) + k^2 \lambda_1^2 \right]^{\frac{1}{2}}$$

which for $h\nu$ small yields

$$\tau \approx \tan^{-1} \left[\frac{k^2 \nu^2}{K} + k^2 \lambda_1^2 \right]^{\frac{1}{2}} \quad \text{which for}$$

k small gives

$$\tau \approx k \sqrt{\frac{2\nu^2}{K} + \lambda_1^2}$$

Thus $e^{\pm in\tau} \approx \exp \left(\pm ix_n \sqrt{\frac{2\nu^2}{K} + \lambda_1^2} \right)$, since $nk = x_n$. This is the same as the second factor

of equation (A-10).

Finally, let us consider the factor $\cos \frac{n}{\tau}$ in equation (A-8). We have

$$\begin{aligned}
\cos^n \tau &= \exp(n \log \cos \tau) \\
&= \exp \left\{ -\frac{n}{2} \log \left[1 + 4p^2 \sin^2 \left(\frac{h\nu}{2} \right) + K^2 \lambda_1^2 \right] \right\} \\
&\approx \exp \left\{ -\frac{n}{2} \log \left[1 + k^2 \left(\frac{\nu^2}{K} + \lambda_1^2 \right) \right] \right\} \quad \text{for small } h\nu \\
&\approx \exp \left\{ -nk \cdot \frac{k}{2} \left(\frac{\nu^2}{K} + \lambda_1^2 \right) \right\} \quad \text{for small } k
\end{aligned}$$

or,

$$\cos^n \tau \approx \exp \left[-\frac{k}{2} \left(\frac{\nu^2}{K} + \lambda_1^2 \right) x_n \right]$$

Thus the solutions in equation (A-8) have damped oscillations. Note that the damping is greater for higher values of frequency ν in the solution and higher values of reduced frequency λ_1 in the differential equation.

APPENDIX B

A STABLE DIFFERENCING SCHEME FOR THE KLEIN-GORDON FORM OF THE FLAT PLATE EQUATION IN SUPERSONIC FLOW

We now establish a difference scheme for equation (A-1) and show that it is stable without introducing attenuation. As before, we suppose a uniform discretization in x and y such that $x_n = nk$ and $y_m = mh$ and denote the value of ψ at (x_n, y_m) by ψ_{nm}

The form of the difference equations to advance the solution from x_n to x_{n+1} are obtained by the following substitutions:

$$\psi_{xx}(x_n, y_m) \rightarrow (\psi_{n+1,m} - 2\psi_{nm} + \psi_{n-1,m})/k^2 \quad (B-1)$$

$$\begin{aligned} \psi_{yy}(x_n, y_m) \rightarrow \frac{1}{h^2} \Big[& a(\psi_{n+1,m+1} - 2\psi_{n+1,m} + \psi_{n+1,m-1}) \\ & + (1-2a)(\psi_{n,m+1} - 2\psi_{n,m} + \psi_{n,m-1}) + a(\psi_{n-1,m+1} - 2\psi_{n-1,m} + \psi_{n-1,m-1}) \Big] \end{aligned} \quad (B-2)$$

where a is a parameter, $a > 0$, to be determined. Making these substitutions into equation (B-1) and multiplying by k^2 , we have the implicit difference equation:

$$\begin{aligned} p^2 \Big[& a(\psi_{n+1,m+1} - 2\psi_{n+1,m} + \psi_{n+1,m-1}) + (1-2a)(\psi_{n,m+1} - 2\psi_{nm} + \psi_{n,m-1}) \\ & + a(\psi_{n-1,m+1} - 2\psi_{n-1,m} + \psi_{n-1,m-1}) \Big] - c_1^2 \psi_{nm} = \psi_{n+1,m} - 2\psi_{nm} + \psi_{n-1,m} \end{aligned} \quad (B-3)$$

where $p^2 = \frac{k^2}{Kh^2}$ and $c_1^2 = k^2 \lambda_1^2$

The parameter a is to be determined from a Von Neumann stability analysis, which we now perform. On substitution into equation (B-3) of $\psi_{nm} = e^{in\theta} e^{im\alpha}$ and subsequent division by $e^{in\theta} e^{im\alpha}$ we obtain

$$p^2 \Big[a(e^{i\theta} - 2 + e^{-i\theta}) + (1-2a)(e^{-i\theta} - 2 + e^{i\theta}) + ae^{-i\alpha}(e^{i\theta} - 2 + e^{-i\theta}) \Big] - c_1^2 = e^{i\alpha} - 2 + e^{-i\alpha}$$

After using the identity

$$e^{ix} - 2 + e^{-ix} = -4 \sin^2(x/2)$$

the preceding equation simplifies to

$$p^2 \Big[-4 \sin^2(\theta/2) \Big] \Big[1 - 4a \sin^2(\alpha/2) \Big] - c_1^2 = -4 \sin^2(\alpha/2)$$

Solving this equation for $\sin^2(\alpha/2)$, we have

$$\sin^2(\alpha/2) = \frac{p^2 \sin^2(\theta/2) + c_1^2 / 4}{1 + 4p^2 a \sin^2(\theta/2)} \quad (B-4)$$

A necessary condition for stability is that equation (B-4) can be solved for real α for every

real θ . This will be true if and only if

$$0 \leq \frac{p^2 \sin^2(\theta/2) + c_1^2/4}{1 + 4p^2 a \sin^2(\theta/2)} \leq 1 \quad (\text{B-5})$$

The left-hand side inequality is automatically satisfied. The right-hand side inequality is equivalent to

$$p^2 [\sin^2(\theta/2)] [1 - 4a] \leq 1 - c_1^2/4 \quad \text{for all } \theta.$$

If a is chosen such that $a \geq (1/4)$, then the left-hand side is ≤ 0 for all θ (and p). Since the left-hand side is 0 for $\theta = 0$ we must have

$$1 - c_1^2/4 = 0 \quad (\text{B-6})$$

Using the definition of c_1^2 we have that equation (B-6) is equivalent to

$$k\lambda_1 \leq 2 \quad (\text{B-7})$$

Thus the difference equation (B-3) is stable for all $p = \frac{k}{h\sqrt{K}}$ provided that

$$(i) \quad a \geq 1/4$$

$$(ii) \quad k\lambda_1 \leq 2$$

Choosing a according to (i) for convenience, we can satisfy (ii) by selecting k sufficiently small for the given reduced frequency and Mach number.

With these restrictions on k and a , we now find the solution ψ_{nm} as h and k go to zero. Then, as before, we let

$$\theta = h\nu$$

and

$$\exp(im\theta) = \exp(i\nu y_m)$$

and equation (B-4) becomes

$$\sin^2(\alpha/2) = \frac{\left(\frac{k^2}{Kh^2}\right) \left(\frac{h^2 \nu^2}{4}\right) + \frac{k^2 \lambda_1^2}{4}}{1 + \left(\frac{k^2}{Kh^2}\right) a h^2 \nu^2}$$

Since h and k are small we have

$$\alpha \approx \frac{k \sqrt{\frac{v^2}{K} + \lambda_1^2}}{1 + \frac{akv^2}{K}} = k \sqrt{\frac{v^2}{K} + \lambda_1^2} + O(k^3)$$

The solution ψ_{nm} to the difference equation then becomes

$$\psi_{nm} = e^{im\theta} e^{in\alpha} = \exp(i\nu y_m) \cdot \exp\left(ix_n \sqrt{\frac{v^2}{K} + \lambda_1^2}\right) + O(k^3)$$

We see that this is the exact solution to the difference equation in equation (A-10) and shows no attenuation of the initial value problem as the solution progresses through the mesh. Since a is the order of unity, its value affects only the third-order terms in the grid spacing.

Before choosing a , it is convenient to write equation (B-3) in another form to maintain the generality.

Tridiagonal Form

Here we consider as known all terms whose ψ superscripts are $\leq n$, and as unknown those terms with ψ superscript equal to $n+1$. Thus equation (B-3) becomes

$$\begin{aligned} & -p^2 a (\psi_{n+1,m+1} - \psi_{n+1,m-1}) + (1 + 2p^2 a) \psi_{n+1,m} \\ & = p^2 (1 - 2a) (\psi_{n+1,m+1} + \psi_{n+1,m-1}) + p^2 a (\psi_{n-1,m+1} + \psi_{n-1,m-1}) \\ & + \left[2(1 - p^2 (1 - 2a)) - c_1^2 \right] \psi_{nm} - (1 + 2ap^2) \psi_{n-1,m} \end{aligned} \quad (B-8)$$

which represents a tridiagonal system for each fixed n .

$$a \approx 1/2$$

For this choice of a , equation (B-8) becomes

$$\begin{aligned} & -\frac{p^2}{2} (\psi_{n+1,m+1} + \psi_{n+1,m-1}) + (1 + p^2) \psi_{n+1,m} \\ & = \frac{p^2}{2} (\psi_{n-1,m+1} + \psi_{n-1,m-1}) + (2 - c_1^2) \psi_{nm} - (1 + p^2) \psi_{n-1,m} \end{aligned}$$

For point relaxation this may be written

$$\psi_{n+1,m} = \frac{p}{2(1+p)} \left[\psi_{n-1,m-1} + \psi_{n-1,m+1} + \psi_{n+1,m-1} + \psi_{n+1,m+1} \right] + \left(\frac{2 - c_1^2}{1 + p^2} \right) \psi_{nm} - \psi_{n-1,m}$$

APPENDIX C

EVALUATING THE WAKE INTEGRAL FOR THE DOWNSTREAM BOUNDARY CONDITION

In the quest for a better formulation of the downstream boundary conditions we assumed the unsteady perturbation potential on the downstream boundary plane to be dominated by the flow induced by the doublet sheet shed from the wing trailing edge. Hence, for two-dimensional flow, the velocity potential at a point (x, y) on the vertical downstream boundary is given by

$$\varphi_1(x, y_1) = \frac{\Delta\varphi_{te}}{4i} \int_{x_{te}}^{\infty} \exp[-i\omega(x' - x_{te})] \cdot [\psi_{y'}(x - x', y_1 - y'_1)]_{y'_1=0} dx'$$

(see, for example, the second term of equation (109) in reference 1). The notation is that used in reference 1.

Since ψ has the form

$$\psi = \psi(x - x', y_1 - y'_1)$$

then

$$\frac{\partial \psi}{\partial y'_1} = - \frac{\partial \psi}{\partial y_1}$$

and we obtain

$$\varphi_1 = \frac{i\Delta\varphi_{te}}{4} \int_{x_{te}}^{\infty} \exp[-i\omega(x' - x_{te})] \cdot \psi_{y_1} dx'_1 \quad (C-1)$$

where, from equation (113) of reference 1

$$\psi = \exp[i\lambda_1 M(x - x')] \cdot H_0^{(2)} \left[\lambda_1 \sqrt{(x - x')^2 + y_1^2} \right] \quad (C-2)$$

and $H_0^{(2)}$ is the Hankel function. From (C-2)

$$\psi_{y_1} = \left\{ -\exp[i\lambda_1 M(x - x')] \cdot \lambda_1 y_1 H_1^{(2)} \left[\lambda_1 \sqrt{(x - x')^2 + y_1^2} \right] \right\} / \sqrt{(x - x')^2 + y_1^2}$$

Since $y_1 = \sqrt{K} y$, we have

$$\psi_{y_1} = \left\{ -\exp i\lambda_1 M(x - x') \cdot \lambda_1 \sqrt{K} y_1 H_1^{(2)}(\lambda_1 r) \right\} / r \quad (C-3)$$

where $r = \sqrt{(x - x')^2 + Ky^2}$

In reference 1 on page 61, it was shown that the integral in equation (C-1) resulting from the combination $\varphi_{1x} + i\omega\varphi_1$ can be integrated in closed form. From equation (78) of reference (1) this is seen to be

$$\varphi_{1_x} + i\omega\varphi_1 = \left\{ \frac{-i\Delta\varphi_{te}}{4} \exp\left[i\lambda_1 M(x - x_{te})\right] \cdot \lambda_1 \sqrt{K} y_1 H_1^{(2)}(\lambda_1 r) \right\} / r \quad (C-4)$$

where $r = \sqrt{(x - x_{te})^2 + Ky^2}$.

The jump in potential φ_1 at the trailing edge can be found in terms of values of the perturbation potential at grid points in the neighborhood of the trailing edge. From equation (104) of reference 1, we have for points on the wing

$$\begin{aligned} \Delta\varphi_1 = & \varphi_{i,j_m+1} - \varphi_{i,j_m} - c_{s1}(\varphi_{i,j_m+2} - \varphi_{i,j_m+1}) \\ & - c_{s2}(\varphi_{i,j_m} - \varphi_{i,j_m-1}) - (d_{s1}F_i^{(U)} + d_{s2}F_i^{(L)}) \end{aligned}$$

where the constants are given in equation (105) and $F_i^{(U)}$ and $F_i^{(L)}$ are the boundary conditions on the upper and lower side, respectively. Since for the sake of economy in computing resources for the test we restricted our analysis to steady-state flows without lift, φ_1 is antisymmetric and

$$\varphi_{i,j_m+1} = -\varphi_{i,j_m}, \quad \varphi_{i,j_m+2} = -\varphi_{i,j_m-1} \quad (C-5)$$

Then on the airfoil

$$\Delta\varphi_i = -2\varphi_{i,j_m} + c_{s1}(\varphi_{i,j_m-1} - \varphi_{i,j_m}) + c_{s2}(\varphi_{i,j_m} - \varphi_{i,j_m-1}) + (d_{s1}F_i^{(U)} + d_{s2}F_i^{(L)}) \quad (C-6)$$

$$\Delta\varphi_i = -(c_{s1} + c_{s2} + 2)\varphi_{i,j_m} + (c_{s1} + c_{s2})\varphi_{i,j_m-1} + (d_{s1}F_i^{(U)} + d_{s2}F_i^{(L)})$$

where the constants are given in equation (105) of reference 1. At the trailing edge the Kutta condition requires

$$\Delta\varphi_{1_x} + i\omega\Delta\varphi_1 = 0 \quad (C-7)$$

at $x = x_{i_1}$, from equation (37) of reference 1, we have

$$c_{1i_1}(\Delta\varphi_{1i_1+1} - \Delta\varphi_{1i_1}) + d_{1i_1}(\Delta\varphi_{1i_1} - \Delta\varphi_{1i_1-1}) + i\omega\Delta\varphi_{1i_1} = 0$$

where c_{1i_1} and d_{1i_1} are given on page 40 of reference 1. Solving for $\Delta\varphi_{1i_1+1}$ yields

$$\Delta\varphi_{1te} = \Delta\varphi_{1i_1+1} = \Delta\varphi_{1i_1} \cdot \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1}\right) + (d_{1i_1} / c_{1i_1})\Delta\varphi_{1i_1-1} \quad (C-8)$$

Using equation (C-6) to define $\Delta\varphi_{i_1}$ and $\Delta\varphi_{i_1-1}$ yields

$$\begin{aligned}
\Delta\varphi_{te} = & \left[1 - \left(d_{1i_1} + i\omega \right) / c_{1i_1} \right] \left[- \left(c_{s1} + c_{s2} + 2 \right) \varphi_{i_1, j_m} \right. \\
& \left. + \left(c_{s1} + c_{s2} \right) \varphi_{i_1, j_m-1} - \left(d_{s1} F_i^{(U)} + d_{s2} F_{i_1}^{(L)} \right) \right] \\
& + \left(d_{1i_1} / c_{1i_1} \right) \left[- \left(c_{s1} + c_{s2} + 2 \right) \varphi_{i_1-1, j_m} + \left(c_{s1} + c_{s2} \right) \varphi_{i_1-1, j_m-1} - \left(d_{s1} F_{i_1-1}^{(U)} + d_{s2} F_{i_1-1}^{(L)} \right) \right]
\end{aligned} \tag{C-8}$$

Hence we write

$$\Delta\varphi_{1te} = h_1 \varphi_{i_1, j_m} + h_2 \varphi_{i_1, j_m-1} + h_3 \varphi_{i_1-1, j_m} + h_4 \varphi_{i_1-1, j_m-1} + R \tag{C-9}$$

where

$$h_1 = - \left(c_{s1} + c_{s2} + 2 \right) \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1} \right) \tag{C-10}$$

$$h_2 = \left(c_{s1} + c_{s2} \right) \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1} \right)$$

$$h_3 = - \left(c_{s1} + c_{s2} + 2 \right) \left(d_{1i_1} / c_{1i_1} \right)$$

$$h_4 = \left(c_{s1} + c_{s2} \right) \left(d_{1i_1} / c_{1i_1} \right)$$

$$\begin{aligned}
R = & - \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1} \right) \left(d_{s1} F_{i_1}^{(U)} + d_{s2} F_{i_1}^{(L)} \right) \\
& - \left(d_{1i_1} / c_{1i_1} \right) \left(d_{s1} F_{i_1-1}^{(U)} + d_{s2} F_{i_1-1}^{(L)} \right)
\end{aligned}$$

We now apply equations (C-4), (C-9), and (C-10) as the boundary conditions on the downstream boundary. Thus in difference form we write

$$\begin{aligned}
& \frac{\varphi_{i_{\max}, j} - \varphi_{i_{\max}-1, j}}{x_{i_{\max}} - x_{i_{\max}-1}} + i\omega \frac{\varphi_{i_{\max}, j} + \varphi_{i_{\max}-1, j}}{2} \\
& = \frac{-i\Delta\varphi_{1te}}{4} \left\{ \exp \left[i\lambda_1 M(a_2 - x_{te}) \right] \right\} \frac{\lambda_1 \sqrt{K} y_j}{r_j} H_1^{(2)}(\lambda_1 r_j)
\end{aligned}$$

where $a_2 = (x_{i_{\max}} + x_{i_{\max}-1}) / 2$ and $r_j = \sqrt{(x_{te} - a_2)^2 + Ky_j^2}$.

This has the same form as equations (119) of reference 1 with a simpler function replacing $P_{i_{\max}j}$. Solving the previous equation for $\Delta\varphi_{1te}$ yields

$$\varphi_{1i_{\max}j} (1 + i\omega\delta_2 / 2) - (1 - i\omega\delta_2 / 2) \varphi_{1i_{\max}-1,j} = \delta_2 \Delta\varphi_{1te} F_j \quad (C-11)$$

where $\delta_2 = x_{i_{\max}} - x_{i_{\max}-1}$ and

$$F_j = -i \left\{ \exp[i\lambda_1 M(a_2 - x_{te})] \cdot \lambda_1 \sqrt{K} y_j H_1^{(2)}(\lambda_1 r_j) \right\} / (4r_j) \quad (C-12)$$

Then

$$\varphi_{i_{\max}j} = c_{k3} \varphi_{i_{\max}-1,j} + c_{k4} \Delta\varphi_{1te} F_j \quad (C-13)$$

where

$$c_{k3} = (1 - i\omega\delta_2 / 2) / (1 + i\omega\delta_2 / 2) \quad (C-14)$$

$$c_{k4} = \delta_2 / (1 + i\omega\delta_2 / 2) \quad (C-15)$$

Substituting for $\Delta\varphi_{1te}$ yields

$$\begin{aligned} \varphi_{i_{\max}j} = & c_{k3} \varphi_{i_{\max}-1,j} \\ & + c_{k4} F_j \left[h_1 \varphi_{i_1,j_m} + h_2 \varphi_{i_1,j_m-1} + h_3 \varphi_{i_1-1,j_m} + h_4 \varphi_{i_1-1,j_m-1} + R \right] \end{aligned} \quad (C-16)$$

In the difference equation for general i,j , the potential $\varphi_{1i_{\max}-1}$ is replaced by the right-hand side of equation (C-16) when $i = i_{\max}-1$, the x index of the downstream boundary plane.

APPENDIX D

A PANEL METHOD FOR MATCHING THE OUTER SOLUTION WITH THE INNER FINITE DIFFERENCE SOLUTION FOR TWO-DIMENSIONAL TRANSONIC UNSTEADY FLOW

D.1 INTRODUCTION

The velocity potential for the unsteady linearized harmonic flow produced by a source distribution on a line segment s is given by

$$\varphi_1 = - \frac{e^{i\lambda_1 Mx}}{4i} \int_s \sigma(x') H_0^{(2)}(\xi) ds \quad (D-1)$$

where $\xi = \lambda_1 r$, $r = \sqrt{(x - x')^2 + K(y - y')^2}$. The derivatives take the form

$$\varphi_{1y} = \frac{\lambda_1^2 K}{4i} \int_s \sigma(x') [H_1^{(2)}(\xi) / \xi] (y - y') ds \left\{ e^{i\lambda_1 Mx} \right. \quad (D-2)$$

$$\varphi_{1x} = \frac{\lambda_1^2}{4i} \int_s \sigma(x') [H_1^{(2)}(\xi) / \xi] (x - x') ds \left\{ e^{i\lambda_1 Mx} + i\lambda_1 M\varphi_1 \right.$$

For convenience, we shall introduce the cylinder functions

$$\ell_n(u) = \left(\frac{2}{\xi}\right)^n H_n^{(2)}(\xi) \quad (D-3)$$

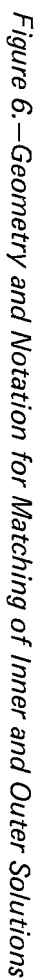
where $u = (\xi / 2)^2 = \frac{\lambda_1^2}{4} [(x - x')^2 + K(y - y')^2]$. The derivatives of the functions take the simple form

$$\ell'_n(u) = -\ell_{n+1}(u) \quad (D-4)$$

and higher order derivatives are obtained by simple recursion formula derived from the differential equation.

$$\ell_n^{k+2}(u) = - \left[(n + k + 1) \ell_n^{k+1}(u) + \ell_n^k(u) \right] / u \quad k \geq 0 \quad (D-5)$$

To match the outer mesh boundary with the proper outgoing wave solution for the rectangular mesh in figure 6, we prescribe the following source and doublet distribution.



$$\begin{aligned}
\varphi_1 e^{-i\lambda_1 Mx} &= \frac{1}{4i} \int \sigma \ell_0(u') ds \\
&= -\frac{1}{4i} \int_{a_1}^{a_2} \left[\sigma_u(x') \ell_0(u'_u) - \sigma_d(x') \ell_0(u'_d) \right] dx' \\
&\quad - \frac{1}{4i} \int_{-b}^{+b} \left[\sigma_\ell(y') \ell_0(u'_\ell) - \sigma_r(y') \ell_0(u'_r) \right] dy' \\
&\quad - \frac{\lambda_1^2 \nabla K y \Delta \varphi_{i_1+1}}{8i} \int_{x_{i_1+1}}^{x''} e^{i\omega(x' - x_{i_1+1})} \ell_1(u'_0) dx'
\end{aligned} \tag{D-6}$$

where $\sigma_u, \sigma_d, \sigma_\ell, \sigma_r$ denote the source strength on the upper, lower, left, and right edges of the mesh region, respectively. The subscript i_1+1 denotes the point just downstream of the trailing edge. Accordingly, the u variables are defined by

$$\begin{aligned}
u'_u &= \frac{\lambda_1^2}{4} \left[(x - x')^2 + K(y - b)^2 \right] \\
u'_d &= \frac{\lambda_1^2}{4} \left[(x - x')^2 + K(y + b)^2 \right] \\
u'_\ell &= \frac{\lambda_1^2}{4} \left[(x - a_1)^2 + K(y - y')^2 \right] \\
u'_r &= \frac{\lambda_1^2}{4} \left[(x - a_2)^2 + K(y - y')^2 \right] \\
u'_0 &= \frac{\lambda_1^2}{4} \left[(x - x')^2 + Ky^2 \right]
\end{aligned} \tag{D-7}$$

To simplify the derivation we assume a symmetric configuration without lift in the steady flow. Then the perturbation potential φ_1 for the unsteady flow is antisymmetric and

$$\begin{aligned}\sigma_\ell(y') &= -\sigma_\ell(-y') \\ \sigma_r(y') &= -\sigma_r(-y')\end{aligned}\tag{D-8}$$

Since we consider only the lower half plane we have

$$\int_{-b}^b \sigma_\ell(y') \ell_0(u'_\ell) dy' = \int_{-b}^0 \sigma_\ell(y') [\ell_0(u_\ell^-) - \ell_0(u_\ell^+)] dy' \tag{D-9}$$

where

$$u_\ell^\pm = \frac{\lambda_1^2}{4} [(x - a_1)^2 + K(y \pm y')^2]$$

Similar relations hold for the right boundary with a_2 replacing a_1 . Finally, for φ_1 :

$$\begin{aligned}\varphi_1 e^{-i\lambda_1 Mx} &= -\frac{1}{4i} \int_{a_1}^{a_2} \sigma_d(x') [\ell_0(u_d) - \ell_0(u_u)] dx' \\ &\quad - \frac{1}{4i} \left\{ \int_{-b}^0 \sigma_\ell(y') [\ell_0(u_\ell^+) - \ell_0(u_\ell^-)] dy' \right. \\ &\quad \left. + \int_{-b}^0 \sigma_r(y') [\ell_0(u_r^+) - \ell_0(u_r^-)] dy' \right\} \\ &\quad + \varphi_w e^{-i\lambda_1 Mx}\end{aligned}\tag{D-10}$$

where we have changed the sign on σ_r for convenience and φ_w is the contribution from the doublet wake:

$$\varphi_w e^{-i\lambda_1 Mx} = -\frac{\lambda_1 \sqrt{K} y \Delta \varphi_{i_1+1}}{8i} \int_{x_{i_1+1}}^{\infty} e^{i\omega(x' - x_{i_1+1})} \ell_1(u'_0) dx' \tag{D-11}$$

where

$$u'_0 = \frac{\lambda_1^2}{4} [(x - x')^2 + Ky^2]$$

Note that equation (D-10) satisfies the requirement $\varphi_1(y) = -\varphi_1(-y)$

D.2 BASIS FUNCTION FOR THE FINITE ELEMENT METHOD

For each station x_i on the upper and lower boundaries we use a linear distribution of doublet strength with the source strength defined at these grid points. For the basis function centered at $x = x_n$, we follow Chen, Dickson, and Rubbert (ref. 9) and use

$$\begin{aligned}\sigma / \sigma_n &= \frac{2(x' - x_{n-1})}{(\delta_{n+1} + \delta_n)\delta_n} \quad \text{for } x_{n-1} \leq x' \leq y_n \\ &= \frac{2(x_{n+1} - x')}{(\delta_{n+1} + \delta_n)\delta_{n+1}} \quad \text{for } x_n \leq x' \leq y_{n+1}\end{aligned}\tag{D-12}$$

where $\delta_k = x_k - x_{k-1}$. This form was chosen so that

$$\int_{x_{n-1}}^{x_{n+1}} [\sigma(x') / \sigma_n] dx' = 1$$

A similar relation holds for sources at station y_n on the $x = a_1$ and $x = a_2$ boundaries. Instead of x_1 and $x_{i_{\max}}$ we replace these values by

$$a_1 = \frac{x_1 + x_2}{2} \quad \text{and} \quad a_2 = \frac{x_{i_{\max}} + x_{i_{\max}-1}}{2}$$

respectively in the end basis functions. We treat the end points on the other boundaries the same way.

We now consider the perturbation velocity potential without the factor $e^{i\lambda_1 Mx}$. Then with

$$\varphi_1 = e^{i\lambda_1 Mx} \psi\tag{D-13}$$

we consider the contribution from the source distributions

$$\psi = \psi_d + \psi_\ell + \psi_r\tag{D-14}$$

where ψ_α is the contribution from the integral over the lower boundary and ψ_ℓ and ψ_r are the contribution from the left and right boundaries, respectively. Substituting the basis function into the integrals and performing the integrations yield

$$\begin{aligned}\psi_d &= \sum_{n=2}^{i_{\max}-1} \sigma_{dn} \psi_{dn} \\ \psi_\ell &= \sum_{n=2}^{j_m} \sigma_{\ell n} \psi_{\ell n} \\ \psi_r &= \sum_{n=2}^{j_m} \sigma_{rn} \psi_{rn}\end{aligned}\tag{D-15}$$

where

$$\begin{aligned}
\psi_{dn} = & -\frac{1}{4i} \frac{2}{(\delta_{n+1} + \delta_n)} \left\{ \int_{x_{n-1}}^{x_n} (x' - x_{n-1}) [\ell_0(u'_d) - \ell_0(u'_u)] dx' / \delta_n \right. \\
& \left. + \int_{x_n}^{x_{n+1}} (x_{n+1} - x') [\ell_0(u'_d) - \ell_0(u'_u)] dx' / \delta_{n+1} \right\} \quad n = 2, \dots, i_{\max}-1 \\
\psi_{\ell n} = & -\frac{1}{4i} \frac{2}{(\tilde{\delta}_{n+1} + \tilde{\delta}_n)} \left\{ \int_{y_{n-1}}^{y_n} (y' - y_{n-1}) [\ell_0(u_{\ell}^+) - \ell_0(u_{\ell}^-)] dy' / \tilde{\delta}_n \right. \\
& \left. + \int_{-y_n}^{y_{n+1}} (y_{n+1} - y') [\ell_0(u_{\ell}^+) - \ell_0(u_{\ell}^-)] dy' / \tilde{\delta}_{n+1} \right\}
\end{aligned} \tag{D-16}$$

where $\tilde{\delta}_k = y_k - y_{k-2}$, $k = 2, 3, \dots, j_m$, and a similar relation holds for ψ_m from the source distribution on the boundary $x = a_2$. We note that, from equation (D-10), the potentials all have the form

$$\psi_t = \psi(y) - \psi(-y) \tag{D-17}$$

The integrals in equation (D-16) may be calculated by Simpson's rule, requiring the evaluation of the $\ell_0(u)$ functions at five points for each integral. With efficient coding this required evaluation of the cylinder functions at the mesh boundary points and at midpoints between them for each point the induced flow is to be calculated. Far-field and near-field expansions of the integrals also may be used to reduce computing costs.

D.3 FORMULAE FOR A FAR-FIELD EXPANSION OF THE INTEGRALS

When the distance from the center of the panel inducing the flow to the point x, y is large compared with the range of integration x_{n-1} to x_{n+1} (or y_{n-1} to y_{n+1}), then the functions $\ell_n(u)$ may be approximated by an expansion of the form

$$\ell_n(u + \Delta u) = \sum_{k=0}^{\infty} \ell_n^{(k)}(u) \cdot \Delta u^k / k! = \sum_{k=0}^{\infty} a_{nk} \Delta u^k \tag{D-18}$$

where u depends only upon x, y and points (x_n, b) , (a_1, y_n) or (a_2, y_n) . We see immediately that

$$a_{n0} = \ell_n(u) \tag{D-19}$$

$$a_{n1} = \ell'_n(u) = -\ell_{n+1}(u)$$

From equations (D-5) and (D-18) we see that the a_{nk} satisfy the recursion relation

$$a_{n,k+2} = - \left[(n+k+1)a_{n,k+1} + \frac{a_{nk}}{k+1} \right] / [(k+2)u]$$

or more conveniently

$$a_{nk} = - \left[(n+k-1)a_{n,k-1} + \frac{a_{n,k-2}}{k-1} \right] / (ku) \quad (D-20)$$

We need to evaluate the integration along x' . Thus we write

$$\begin{aligned} \ell_0(u') &= \ell_0(u + \Delta u) \\ &= \sum_{k=0}^{\infty} a_{0k} \Delta u^k \end{aligned} \quad (D-21)$$

where

$$\begin{aligned} a_{00} &= \ell_0(u) \\ a_{01} &= -\ell_1(u) \\ a_{0k} &= - \left[(k-1)a_{0k-1} + a_{0k-2} / (k-1) \right] / (ku) \end{aligned} \quad (D-22)$$

Substituting equation (D-21) into equation (D-16) yields

$$\begin{aligned} \psi_{dn} &= - \frac{1}{2i(\delta_{n+1} + \delta_n)} \sum_{k=0}^{\infty} a_{0k} \left\{ \int_{x_{n-1}}^{x_n} (x' - x_{n-1}) \Delta u^k dx' / \delta_n \right. \\ &\quad \left. + \int_{x_n}^{x_{n+1}} (x_{n+1} - x') \Delta u^k dx' / \delta_{n+1} \right\} \end{aligned} \quad (D-23)$$

where, as we shall see, $\Delta u = \mu(w_1 - x')(w_2 - x')$. For convenience we shall introduce the functions

$$U_{nk}(x') = \int_{x'}^{x_n} \Delta u^k dx' \quad (D-24)$$

Note that equation (D-23) contains only one of the functions in equation (D-16) for the sake of simplicity.

We require the functions U_{0k} and U_{1k} , and for later considerations, U_{2k} . For the lower boundary we choose

$$u_d' = u_d + \Delta u$$

where

$$\begin{aligned} u_d' &= \frac{\lambda_1^2}{4} \left[(x - x')^2 + K(y + b) \right] \\ u_d &= \frac{\lambda_1^2}{4} \left[(x - x_n)^2 + K(y + b) \right] \end{aligned} \quad (D-25)$$

then

$$\Delta u = \frac{\lambda_1^2}{4} \left[(x - x')^2 - (x - x_n)^2 \right] \quad (D-26)$$

$$\Delta u = \mu \left[(w_1 - x')(w_2 - x') \right]$$

where

$$w_1 = x_n, w_2 = 2x - x_n, \text{ and } \mu = \lambda_1^2 / 4 \quad (D-27)$$

Similarly, for the left-hand boundary we have

$$u'_Q = u_Q + \Delta u$$

where

$$u'_Q = \frac{\lambda_1^2}{4} \left[(x - a_1)^2 + K(y - y')^2 \right] \quad (D-28)$$

$$u_Q = \frac{\lambda_1^2}{4} \left[(x - a_1)^2 + K(y - y_n)^2 \right]$$

from which

$$\Delta u = \frac{\lambda_1^2 K}{4} \left[(w_1 - y')(w_2 - y') \right]$$

where $w_1 = y_n$ and $w_2 = 2y - y_n$. If we define $\mu = \frac{\lambda_1^2 K}{4}$ then

$$\begin{aligned} \Delta u &= \mu \left[(w_1 - y')(w_2 - y') \right] \\ \Delta u &= \mu \left[(w_1 - x')(w_2 - x') \right] \end{aligned} \quad (D-29)$$

and the integrals along the boundaries take the same form.

We now evaluate the functions $U_{jk}(x')$. Thus with equation (D-29)

$$U_{0k}(x') = \int \mu^k (w_1 - x')^k (w_2 - x')^k dx' \quad (D-30)$$

Let $w_1 - x' = \xi$; then

$$U_{0k}(x') = -\mu^k \int \xi^k (w_2 - w_1 + \xi)^k d\xi \quad (D-31)$$

Expanding the k^{th} power of the term in parenthesis and integrating yields

$$U_{0k}(x') = -\mu^k \sum_{j=0}^k \binom{k}{j} (w_2 - w_1)^{k-j} (w_1 - x')^{k+j+1} / (k+j+1) \quad (D-32)$$

where $\binom{k}{j}$ are Newton's binomial coefficients.

Writing

$$\begin{aligned}\mu(w_1 - x')(w_2 - w_1) &= \alpha \\ (w_1 - x') / (w_2 - w_1) &= t\end{aligned}\tag{D-33}$$

yields

$$U_{0k}(x') = -\alpha^k (w_1 - x') \sum_{j=0}^k \binom{k}{j} \frac{t^j}{k+j+1}\tag{D-34}$$

Now

$$\begin{aligned}U_{1k}(x') &= \int \mu^k (w_1 - x')^k (w_2 - x')^k x' dx' \\ &= w_1 U_{0k} + \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(w_2 - w_1)^{k-j} (w_1 - x')^{k+j+2}}{k+j+2} \\ &= w_1 U_{0k} + \alpha^k (w_1 - x')^2 \sum_{j=0}^k \binom{k}{j} \frac{t^j}{k+j+2}\end{aligned}\tag{D-35}$$

Similarly

$$\begin{aligned}U_{2k}(x') &= -\int \mu^k \xi^k (w_2 - w_1 + \xi)^k (w_1 - \xi)^2 d\xi \\ &= w_1^2 U_{0k} + 2w_1 \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(w_2 - w_1)^{k-j} (w_1 - x')^{k+j+2}}{k+j+2} \\ &\quad - \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(w_2 - w_1)^{k-j} (w_1 - x')^{k+j+3}}{k+j+3}\end{aligned}\tag{D-36}$$

Note that $w_1 - x' = 0$ for $x' = x_n$ and $w_1 = w_2$ for $x = x_n$. For this special case the U_{nk} take a simpler form

$$\begin{aligned}U_{0k} &= \int \mu^k (w_1 - x')^{2k} dx' \\ &= -\mu^k (w_1 - x')^{2k+1} / (2k+1)\end{aligned}\tag{D-37}$$

$$U_{1k} = w_1 U_{0k} + \mu^k (w_1 - x')^{2k+2} / (2k+2)\tag{D-38}$$

$$U_{2k} = -w_1^2 U_{0k} + 2w_1 U_{1k} - \mu^k (w_1 - x')^{2k+3} / (2k+3)\tag{D-39}$$

Denoting

$$U_{mk}(x_n) = U_{mk}^n \quad (D-40)$$

and substituting for U_{nk} into the right-hand side of equation (D-23) yields

$$-\frac{a_{00}}{4i} - \frac{1}{2i(\delta_{n+1} + \delta_n)} - \sum_{k=1}^{\infty} \left\{ \left[x_{n-1} U_{0k}^{n-1} - U_{1k}^{n-1} \right] / \delta_n \right. \quad (D-41)$$

$$\left. + \left[x_{n+1} U_{0k}^{n+1} - U_{1k}^{n+1} \right] / \delta_{n+1} \right\} a_{0k}$$

Since the a_{0k} are functions of u , which we shall define for convenience in the form

$$u = \mu \left[(x - x_n)^2 + z^2 \right] \quad (D-42)$$

and since the U_{nk} are functions of μ , $w_1 = x_n$, $w_2 = 2x - x_n$, we can define a general function

$$G_0(\mu, x, z, x_{n-1}, x_n, x_{n+1}) \quad (D-43)$$

$$= -\frac{a_{00}}{4i} - \frac{1}{2i(\delta_{n+1} + \delta_n)} \sum_{k=1}^{\infty} \left\{ \left[x_{n-1} U_{0k}^{n-1} - U_{1k}^{n-1} \right] / \delta_n \right.$$

$$\left. + \left[x_{n+1} U_{0k}^{n+1} - U_{1k}^{n+1} \right] / \delta_{n+1} \right\} a_{0k}$$

Then we have finally

$$\psi_{dn} = G_0 \left(\frac{\lambda_1^2}{4}, x, \sqrt{K} (y + b), x_{n-1}, x_n, x_{n+1} \right) \quad (D-44)$$

$$\psi_{\ell n} = G_0 \left(\frac{\lambda_1^2 K}{4}, y, (x - a_1) / \sqrt{K}, y_{n-1}, y_n, y_{n+1} \right)$$

$$\psi_m = G_0 \left(\frac{\lambda_1^2 K}{4}, y, (x - a_2) / \sqrt{K}, y_{n-1}, y_n, y_{n+1} \right)$$

D.4 FORMULAE FOR THE NEAR-FIELD EXPANSION OF THE INTEGRALS

For the near field, the argument of u in the $\ell_n(u)$ functions is assumed sufficiently small that the power series of the functions may be integrated term by term. Now

$$J_n(\xi) = \left(\frac{\xi}{2} \right)^n \sum_{k=0}^{\infty} \frac{(\xi/2)^{2k} (-1)^k}{k!(n+k)!}$$

$$Y_n(\xi) = -\frac{1}{\pi} \left(\frac{2}{\xi}\right)^n \sum_{k=0}^{n-1} \frac{(n-k-1)}{k!} \left(\frac{\xi}{2}\right)^{2k} + \frac{2}{\pi} \log\left(\frac{\xi}{2}\right) J_n(\xi) \quad (D-45)$$

$$- \frac{1}{\pi} \left(\frac{\xi}{2}\right)^n \sum_{k=0}^{\infty} \left\{ \psi(k+1) + \psi(k+n+1) \right\} \frac{(-1)^k (\xi/2)^{2k}}{k!(n+k)!} \quad (D-46)$$

where $\psi(1) = \gamma$ and $\psi_n = -\gamma + \sum_{m=1}^{n-1} \ell/m$ Since $u = (\xi/2)^2$ then

$$\ell_0^J(u) = J_0(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k u^k}{(k!)^2} = \sum_{k=0}^{\infty} c_{0k} u^k \quad (D-47)$$

Similarly

$$\ell_1^J(u) = 2J_1(\xi)/\xi = \sum_{k=0}^{\infty} \frac{(-1)^k u^k}{k!(k+1)!} = \sum_{k=0}^{\infty} c_{1k} u^k \quad (D-48)$$

$$\text{where } c_{0k} = (-1)^k / (k!)^2 \text{ and } c_{1k} = c_{0k} / (k+1) \quad (D-49)$$

We also have

$$\ell_0^Y(u) = Y_0(\xi) = \frac{1}{\pi} \log u \ell_0^J(u) - \frac{2}{\pi} \sum_{k=0}^{\infty} \psi(k+1) c_{0k} u^k \quad (D-50)$$

$$\ell_0^Y(u) = Y_0(\xi) = \frac{1}{\pi} \log u \ell_0^J(u) - \frac{2}{\pi} \sum_{k=0}^{\infty} \psi(k+1) c_{0k} u^k \quad (D-51)$$

$$\ell_1^Y(u) = 2Y_1(\xi)/\xi = -\frac{1}{\pi u} + \frac{1}{\pi} \log u \ell_1^J(u)$$

$$- \frac{1}{\pi} \sum_{k=0}^{\infty} \left\{ \psi(k+1) + \psi(k+2) \right\} c_{1k} u^k$$

Since $\ell_0(u) = H_0^{(2)}(\xi) = \ell_0^J(u) - i\ell_0^Y(u)$, then

$$\ell_0(u) = \sum_{k=0}^{\infty} \left\{ b_{0k} - \frac{ic_{0k}}{\pi} \log u \right\} u^k \quad (D-52)$$

where

$$b_{0k} = c_{0k} \left[1 + \frac{2i}{\pi} \psi(k+1) \right] \quad (D-53)$$

Similarly

$$\ell_1(u) = \sum_{k=0}^{\infty} \left[b_{1k} - \frac{ic_{1k}}{\pi} \log u \right] u^k + \frac{i}{\pi u} \quad (D-54)$$

where

$$b_{1k} = c_{1k} \left\{ 1 + \frac{i}{\pi} [\psi(k+1) + \psi(k+2)] \right\} \quad (D-55)$$

We need to evaluate integrals of the form

$$V_{nk}(x') = \int x'^n u^k dx'$$

and

$$W_{nk}(x') = \int x'^n u^k \log u dx' \quad (D-56)$$

For integration along x' , we have

$$u = \frac{\lambda_1^2}{4} [(x - x')^2 + K(y + b)^2]$$

We now factor the quadratic in a form similar to equation (D-29). This leads to

$$u = \frac{\lambda_1^2}{4} (w_0 - x')(\bar{w}_0 - x')$$

where $w_0 = x + i(y - b)\sqrt{K}$ and \bar{w}_0 is the complex conjugate. Similarly for integration with respect to y' we have

$$u = \frac{\lambda_1^2 K}{4} [(w_1 - y')(\bar{w}_1 - y')]$$

where

$$w_1 = y + i(x - a_1) / \sqrt{K}.$$

As before we define

$$u = \mu [(w - x')(\bar{w} - x')] \quad (D-57)$$

and obtain a general formula that holds, with appropriate arguments, for each of the three boundaries. Thus

$$\begin{aligned} V_{0k} &= \int u^k dx' \\ &= \mu^k \int (w - x')^k (\bar{w} - x')^k dx' \end{aligned} \quad (D-58)$$

Comparing with equations (D-34) and (D-36) we obtain

$$V_{0k} = -\mu^k \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+1}}{k+j+1} \quad (D-59)$$

Similarly from equation (D-35) and (D-36), we have

$$V_{1k} = w V_{0k} + \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+2}}{k+j+2} \quad (D-60)$$

$$V_{2k} = 2wV_{1k} - w^2 V_{1k} - \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+3}}{k+j+3} \quad (D-61)$$

Now

$$\log u = \log \mu + \log(w - x') + \log(\bar{w} - x')$$

then we have

$$\begin{aligned} W_{0k}(x') &= \int u^k \log u \, dx' \\ &= V_{0k} \log \mu + V_{0k} \log(w - x') + \int \frac{V_{0k}(x') dy'}{w - x'} + \bar{V}_{0k} \log(\bar{w} - x') + \int \frac{\bar{V}_{0k}(x') dy'}{\bar{w} - x'} \end{aligned}$$

Substituting for V_{0k} yields

$$\begin{aligned} W_{0k} &= V_{0k} \log \mu + 2 \operatorname{Real} \left\{ V_{0k} \log(w - x') \right. \\ &\quad \left. + \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+1}}{(k+j+1)^2} \right\} \end{aligned} \quad (D-62)$$

Now

$$\begin{aligned} w_{1k}(x') &= \int x' u^k \log u \, dx' \\ &= V_{1k} \log \mu + V_{1k} \log(w - x') + \int \frac{V_{1k}(x') dy'}{w - x'} \\ &\quad + \bar{V}_{1k} \log(\bar{w} - x') + \int \frac{\bar{V}_{1k}(x') dx'}{\bar{w} - x'} \end{aligned}$$

Substituting for V_{0k} yields

$$\begin{aligned} w_{1k}(x') &= V_{1k} \log \mu + 2 \operatorname{Real} \left\{ V_{1k} \log(w - x') \right. \\ &\quad \left. + \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+1}}{(k+j+1)^2} \right. \\ &\quad \left. - \mu^k \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+2}}{(k+j+2)^2} \right\} \end{aligned} \quad (D-63)$$

Similarly

$$\begin{aligned}
w_{2k}(x') = V_{2k} \log \mu + 2 \operatorname{Re} \left\{ V_{2k} \log(w - x') \right. \\
+ w \frac{2}{\mu} \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+1}}{(k+j+1)^2} \\
+ 2w\mu \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+2}}{(k+j+2)^2} \\
\left. - \mu \sum_{j=0}^k \binom{k}{j} \frac{(\bar{w} - w)^{k-j} (w - x')^{k+j+3}}{(k+j+3)^2} \right\}
\end{aligned} \tag{D-64}$$

With the functions V_{jk} and w_{jk} defined and with

$$V_{jk}(x_n) = V_{jk}^n,$$

Substituting equation (D-64) into (D-23) leads to

$$\begin{aligned}
\psi_{dn} = & - \frac{1}{2i(\delta_{n+1} + \delta_n)} \sum_{k=0}^{\infty} \left\{ \int_{x_{n-1}}^{x_n} (x' - x_{n-1}) \left[b_{0k} - \frac{iC_{0k}}{\pi} \log u \right] u^k dx' / \delta_n \right. \\
& \left. + \int_{x_n}^{x_{n+1}} (x_{n+1} - x') \left[b_{0k} - \frac{iC_{0k}}{\pi} \log u \right] u^k dx' / \delta_{n+1} \right\} \\
\psi_{dn} = & - \frac{b_{00}}{4i} - \frac{1}{2i(\delta_{n+1} + \delta_n)} \sum_{k=1}^{\infty} b_{0k} \left\{ \left[x_{n-1} (V_{0k}^{n-1} - V_{0k}^n) \right. \right. \\
& \left. \left. - (V_{1k}^{n-1} - V_{1k}^n) \right] / \delta_n + \left[x_{n+1} (V_{0k}^{n+1} - V_{0k}^n) - (V_{1k}^{n+1} - V_{1k}^n) \right] / \delta_{n+1} \right\} \\
& + \frac{1}{2\pi(\delta_{n+1} + \delta_n)} \sum_{k=0}^{\infty} C_{0k} \left\{ \left[x_{n-1} (w_{0k}^{n-1} - w_{0k}^n) \right. \right. \\
& \left. \left. - (w_{1k}^{n-1} - w_{1k}^n) \right] / \delta_n + \left[x_{n+1} (w_{0k}^{n+1} - w_{0k}^n) \right. \right. \\
& \left. \left. - (w_{1k}^{n+1} - w_{1k}^n) \right] / \delta_{n+1} \right\}
\end{aligned} \tag{D-65}$$

Writing for the variable w

$$w = x + iz$$

and defining the function

$$P_0(\mu, x, z, x_{n-1}, x_n, x_{n+1}) \quad (D-66)$$

as the left-hand side of equation (D-65), we obtain

$$\psi_{dn} = P_0\left(\frac{\lambda_1^2}{4}, x, \sqrt{K}(y+b), x_{n-1}, x_n, x_{n+1}\right) \quad (D-67)$$

$$\psi_{\ell n} = P_0\left(\frac{\lambda_1^2 K}{4}, y, (x - a_1) / \sqrt{K}, y_{n-1}, y_n, y_{n+1}\right)$$

and similar relations for ψ_{un} and ψ_{rn} .

D.5 CALCULATION OF NORMAL DERIVATIVE TO MESH BOUNDARIES

The calculation of the contribution to φ_{1y} on $y = -b$ from the source distribution on the line $y = -b$ is very simple. We consider

$$\psi(x, y) = -\frac{1}{4i} \int_{a_1}^{a_2} \sigma(x') \ell_0(u) dx'$$

$$\text{where } u = \frac{\lambda_1^2}{4} \left[(x - x')^2 + K(y + b)^2 \right] \quad . \text{ Let } \sqrt{K}(y + b) = \eta \quad \text{then } u = \frac{\lambda_1^2}{4} \left[(x - x')^2 + \eta^2 \right]$$

and

$$\psi_y = \sqrt{K} \psi_{1\eta}$$

$$\psi_\eta = \frac{\lambda_1 \eta}{8i} \int_{a_1}^{a_2} \sigma(x') \ell_1(u) dx'$$

Near $u = 0$, $\ell_1(u) = i / \pi u + O(1)$; thus near $\eta = 0$ the integral takes the form

$$\psi_\eta = \frac{\eta}{2\pi} \int_{a_1}^{a_2} \frac{\sigma(x') dx'}{(x - x')^2 + \eta^2}$$

Since the integral does not exist at $\eta = 0$, we introduce the variable

$$(x' - x) / \eta = \xi$$

and obtain

$$\psi_\eta = \frac{1}{2\pi} \int_{(a_1 - x) / \eta}^{(a_2 - x) / \eta} \sigma(x + \eta \xi) d\xi / (\xi^2 + 1)$$

If y goes to $-b$ through values of $y > -b$, then η is positive and for $a_1 < x < a_2$, the limit as η goes to 0 becomes

$$\psi_\eta = \sigma(x) / 2 \quad (D-68)$$

and

$$\psi_y = \sqrt{K}\sigma(x) / 2$$

If y goes to $-b$ through values of $y < -b$, then the limit becomes

$$\psi_y = -\sqrt{K}\sigma(x) / 2 \quad (D-69)$$

Similarly for sources on the y' axis we write

$$\psi = -\frac{1}{4i} \int_{b_1}^{b_2} \sigma(y') \ell_0(u) dy'$$

where $u = \frac{\lambda_1^2}{4} \left[(x - a_1)^2 + K(y - y')^2 \right]$, let $\sqrt{K}(y' - y) = \eta'$, then

$$\psi = -\frac{1}{4i\sqrt{K}} \int_{\sqrt{K}(b_1-y)}^{\sqrt{K}(b_2-y)} \sigma(\eta') \ell_0(u) d\eta'$$

Since this has the same form as the $\sigma(x')$ contribution, we see that

$$\psi_x = \sigma(y) / (2\sqrt{K}) \quad (D-70)$$

for $-b < y < b$ and x going to a_1 through values of $x > a_1$. For x going to a_1 through values of $x < a_1$, we obtain

$$\psi_x = -\sigma(y) / (2\sqrt{K}) \quad (D-71)$$

D.6 THE BOUNDARY CONDITIONS ON THE MESH BOUNDARIES

To match the interior finite difference solution with the outer finite element solution, we make the values of the potential from the two solutions and the values of the normal derivatives from the two solutions equal on the mesh boundary. Thus on $y = -b = (y_1 + y_2) / 2$, we match the values of

$$\varphi_1 \text{ and } \varphi_{1y}$$

from the two solutions while on $x = a_1 = (x_1 + x_2) / 2$ and on $x = a_2 = (x_{i_{\max}} + x_{i_{\max}-1}) / 2$, we match the values of

$$\varphi_1 \text{ and } \varphi_{1x}$$

from the two solutions.

We could actually evaluate φ_x and φ_y for the outer solution by differentiating φ from the wake and source distributions; but to simplify the programming, we will approximate φ_1 at $x = x_i$ by $(\varphi_{i1} + \varphi_{i2}) / 2$ on the lower boundary both for the finite difference solution and for the finite element solution. For the derivative with respect to y , we take

$$\varphi_{1y} = (\varphi_{i2} - \varphi_{i1}) / (y_2 - y_1)$$

for the finite difference solution and from the exterior solutions for the source distributions on the sides $x = a_1$ and $x = a_2$; but we will use

$$\lim_{y \rightarrow -b} \frac{\partial \varphi_d}{\partial y}$$

from the source distribution terms for the lower boundary $y = -b$.

Let $\varphi_l, \varphi_r, \varphi_d, \varphi_w$ be the contributions to the exterior solution from the left, right, lower boundary source distributions, and the wake. Then the boundary conditions on $y = -b$ at $x = x_i$ are

$$\begin{aligned} \frac{\varphi_{i1}^+ + \varphi_{i2}^+}{2} &= \frac{\varphi_{li1}^+ + \varphi_{li2}^+ + \varphi_{ri1}^+ + \varphi_{ri2}^+ + \varphi_{di1}^+ + \varphi_{di2}^+}{2} \\ &\quad - \frac{\varphi_{di1}^- + \varphi_{di2}^-}{2} + \frac{\varphi_{wi1}^+ + \varphi_{wi2}^+}{2} \\ \frac{\varphi_{i1}^- - \varphi_{i2}^-}{y_1 - y_2} &= \frac{\varphi_{li1}^- - \varphi_{li2}^- + \varphi_{ri1}^- - \varphi_{ri2}^-}{(y_1 - y_2)} + \lim_{y \rightarrow -b} \frac{\partial \varphi_d}{\partial y} \\ &\quad - \frac{\varphi_{di1}^- - \varphi_{di2}^-}{y_1 - y_2} + \frac{\varphi_{wi1}^- - \varphi_{wi2}^-}{y_1 - y_2} \end{aligned} \quad (D-72)$$

where the + and - denote the lower source distribution and its image at $y = b$, and

$\varphi_r = \varphi_r^+ - \varphi_r^-$ and $\varphi_l = \varphi_l^+ - \varphi_l^-$. Since we are interested in the outer solution at $y = -b$ through values of $y < -b$, then from the preceding equation we obtain

$$\begin{aligned} \frac{\varphi_{i1}^- - \varphi_{i2}^-}{2} &= \frac{\varphi_{li1}^- - \varphi_{li2}^- + \varphi_{ri1}^- - \varphi_{ri2}^-}{2} \\ &\quad - \sqrt{K} \sigma_{di} e^{i\lambda_1 M x_i} (y_1 - y_2) / 4 - \frac{\varphi_{di1}^- - \varphi_{di2}^-}{2} \end{aligned} \quad (D-73)$$

Adding and subtracting equations (D-72) and (D-73) yields

$$\varphi_{i1} = \varphi_{li1} + \varphi_{ri1} - \varphi_{di1} + \left(\varphi_{di1}^+ + \varphi_{di2}^+ \right) / 2 - \sigma_{di} e^{i\lambda_1 M x_i} (y_1 - y_2) \sqrt{K} / 4 + \varphi_{wi1} \quad (D-74)$$

$$\varphi_{i2} = \varphi_{li2} + \varphi_{ri2} - \varphi_{di2} + \left(\varphi_{di1}^+ + \varphi_{di2}^+ \right) / 2 + \sigma_{di} e^{i\lambda_1 M x_i} (y_1 - y_2) \sqrt{K} / 4 + \varphi_{wi2} \quad (D-75)$$

Because of the factor $e^{i\lambda_1 M x}$, the normal boundary conditions on $x = a_1$ and $x = a_2$ take a different form. For the derivative with respect to x we have

$$\frac{\partial \varphi_l^+}{\partial x} = e^{i\lambda_1 M x} \left[\psi_{lx}^+ + i\lambda_1 M \psi_l^+ \right]$$

Since we must approach $x = a_1$ from values $x < a_1$, we obtain from equation (D-71)

$$\lim_{x \rightarrow a_1^+} \frac{\partial \varphi_\ell}{\partial x} = i\lambda_1 M \varphi_\ell^+ - \sigma_\ell(y) e^{i\lambda_1 Ma_1} / 2\sqrt{K}$$

Thus the boundary conditions on $x = a_1$ become

$$\frac{\varphi_{1j}^+ + \varphi_{2j}^+}{2} = \frac{\varphi_{\ell 1j}^+ + \varphi_{\ell 2j}^+ - \varphi_{\ell 1j}^- - \varphi_{\ell 2j}^- + \varphi_{r 2j}^+ + \varphi_{r 1j}^-}{2} + \frac{\varphi_{d 1j}^+ + \varphi_{d 2j}^+}{2} + \frac{\varphi_{w 1j}^+ + \varphi_{w 2j}^+}{2} \quad (D-76)$$

$$\frac{\varphi_{1j}^- - \varphi_{2j}^-}{x_1 - x_2} = \frac{-\varphi_{\ell 1j}^- + \varphi_{\ell 2j}^- + \varphi_{r 1j}^- - \varphi_{r 2j}^+}{x_1 - x_2} + \frac{\varphi_{w 1j}^- - \varphi_{w 2j}^-}{x_1 - x_2} + \frac{\varphi_{d 1j}^- - \varphi_{d 2j}^-}{x_1 - x_2} - \sigma_\ell(y_j) e^{i\lambda_1 Ma_1} / (2\sqrt{K}) + i\lambda_1 M \left(\frac{\varphi_{\ell 1j}^+ + \varphi_{\ell 2j}^+}{2} \right) \quad (D-77)$$

Adding and subtracting equations (D-76) and $(x_1 - x_2) / 2$ times equation (D-77) yield

$$\varphi_{1j} = \alpha_0 (\varphi_{\ell 1j}^+ + \varphi_{\ell 2j}^+) - \varphi_{\ell 1j}^- + \varphi_{r 1j}^+ + \varphi_{d 1j}^+ + \varphi_{w 1j}^- - \left(\sigma_\ell e^{i\lambda_1 Ma_1} / (4\sqrt{K}) \right) (x_1 - x_2) \quad (D-78)$$

$$\varphi_{2j} = \bar{\alpha}_0 (\varphi_{\ell 1j}^+ + \varphi_{\ell 2j}^+) - \varphi_{\ell 2j}^- + \varphi_{r 2j}^+ + \varphi_{d 2j}^+ + \varphi_{w 2j}^- + \left(\sigma_\ell e^{i\lambda_1 Ma_1} / (4\sqrt{K}) \right) (x_1 - x_2) \quad (D-79)$$

where $\alpha_0 = 1 / 2 + iM\lambda_1(x_1 - x_2) / 4$.

On the right-hand boundary we apply the boundary conditions

$$\frac{(\varphi_{i_{\max}j} - \varphi_{i_{\max}-1,j})}{(x_{i_{\max}} - x_{i_{\max}-1})} = \frac{\varphi_{\ell i_{\max}j} - \varphi_{\ell i_{\max}-1,j} + \varphi_{d i_{\max}j} - \varphi_{d i_{\max}-1,j}}{(x_{i_{\max}} - x_{i_{\max}-1})} \quad (D-80)$$

$$- \frac{\varphi_{r i_{\max}j} - \varphi_{r i_{\max}-1,j}}{x_{i_{\max}} - x_{i_{\max}-1}} + \frac{\varphi_{w i_{\max}j} - \varphi_{w i_{\max}-1,j}}{x_{i_{\max}} - x_{i_{\max}-1}} + \lim_{x \rightarrow a_2} \frac{\partial \varphi_r}{\partial x}$$

$$\frac{\varphi_{i_{\max}j} + \varphi_{i_{\max}-1,j}}{2} = \frac{\varphi_{\ell i_{\max}j} + \varphi_{\ell i_{\max}-1,j} + \varphi_{d i_{\max}j} + \varphi_{d i_{\max}-1,j}}{2} + \frac{\varphi_{r i_{\max}j} + \varphi_{r i_{\max}-1,j}}{2} - \frac{\varphi_{r i_{\max}j}^- + \varphi_{r i_{\max}-1,j}^-}{2} + \frac{\varphi_{w i_{\max}j} + \varphi_{w i_{\max}-1,j}}{2} \quad (D-81)$$

Now

$$\lim_{x \rightarrow a_2} \frac{\partial \varphi_r}{\partial x} = i\lambda_1 M \varphi_r^+ + \lim_{x \rightarrow a_2} e^{i\lambda_1 Ma_2} \frac{\partial \varphi_r}{\partial r}$$

Since we approach a_2 from the outside, or $x > a_2$, we have from equation (D-70)

$$\lim_{x \rightarrow a_2} \frac{\partial \varphi_r}{\partial x} = i\lambda_1 M \varphi_r + e^{i\lambda_1 Ma_2} \sigma_{rj} / (2\sqrt{K}) \quad (D-82)$$

Thus equation (D-80) with the aid of equation (D-82) becomes

$$\begin{aligned} \frac{\varphi_{i_{\max}j} - \varphi_{i_{\max}-1,j}}{x_{i_{\max}} - x_{i_{\max}-1}} &= \frac{\varphi_{\ell i_{\max}j} - \varphi_{\ell i_{\max}-1,j} + \varphi_{di_{\max}j} - \varphi_{di_{\max}-1,j}}{x_{i_{\max}} - x_{i_{\max}-1}} \\ &\quad - \frac{\varphi_{ri_{\max}j} - \varphi_{ri_{\max}-1,j}}{x_{i_{\max}} - x_{i_{\max}-1}} + i\lambda_1 M \frac{\varphi_{ri_{\max}j} + \varphi_{ri_{\max}-1,j}}{2} \\ &\quad + e^{i\lambda_1 Ma_2} \sigma_{rj} / (2\sqrt{K}) + \frac{\varphi_{wi_{\max}j} - \varphi_{wi_{\max}-1,j}}{2} \end{aligned} \quad (D-83)$$

and the other boundary condition is

$$\begin{aligned} \frac{\varphi_{i_{\max}j} + \varphi_{i_{\max}-1,j}}{2} &= \frac{\varphi_{\ell i_{\max}j} + \varphi_{\ell i_{\max}-1,j} + \varphi_{di_{\max}j} + \varphi_{di_{\max}-1,j}}{2} \\ &\quad + \frac{-\varphi_{ri_{\max}j} - \varphi_{ri_{\max}-1,j} + \varphi_{ri_{\max}j} + \varphi_{ri_{\max}-1,j}}{2} \\ &\quad + \frac{\varphi_{wi_{\max}j} + \varphi_{wi_{\max}-1,j}}{2} \end{aligned} \quad (D-84)$$

Adding $(x_{i_{\max}} - x_{i_{\max}-1}) / 2$ times equation (D-83) to equation (D-84) yields

$$\begin{aligned} \varphi_{i_{\max}-1j} &= \varphi_{\ell i_{\max}-1,j} + \varphi_{di_{\max}-1,j} - \varphi_{ri_{\max}-1,j} + \varphi_{wi_{\max}-1,j} \\ &\quad + \alpha_1 \left(\varphi_{ri_{\max}j} + \varphi_{ri_{\max}-1,j} \right) - e^{i\lambda_1 Ma_2} \sigma_{rj} (x_{i_{\max}} - x_{i_{\max}-1}) / (4\sqrt{K}) \end{aligned} \quad (D-85)$$

Subtracting leads to

$$\begin{aligned} \varphi_{i_{\max}j} &= \varphi_{\ell i_{\max}j} + \varphi_{di_{\max}j} - \varphi_{ri_{\max}j} + \varphi_{wi_{\max}j} + \alpha_1 \left(\varphi_{ri_{\max}j} + \varphi_{ri_{\max}-1,j} \right) \\ &\quad + e^{i\lambda_1 Ma_2} \sigma_{rj} (x_{i_{\max}} - x_{i_{\max}-1}) / (4\sqrt{K}) \end{aligned} \quad (D-86)$$

where

$$\alpha_1 = 1 / 2 + i\lambda_1 M (x_{i_{\max}} - x_{i_{\max}-1}) / 4$$

D.7 EVALUATION OF THE WAKE INTEGRAL FOR THE OUTER SOLUTION

We must also include the contributions from the doublet wake integral in the outer solution. From equation (110), page 68, of NASA CR-2257, we have

$$\begin{aligned}\varphi_w &= \frac{i\Delta\varphi_t}{4} \int_{x_{i_1+1}}^{\infty} e^{-i\omega(x'-x_{i_1+1})} \psi_{y_1} dx'_1 \\ &= \frac{i\Delta\varphi_t}{4\sqrt{K}} \int_{x_{i_1+1}}^{\infty} e^{-i\omega(x'-x_{i_1+1})} \psi_y dx'\end{aligned}\quad (D-87)$$

From the bottom of page 68 this takes the form

$$\varphi_w = -\frac{i\lambda_1 \sqrt{K} y}{4} \Delta\varphi_t \left\{ \int_{x_{i_1+1}}^{\infty} e^{-i\omega(x'-x_{i_1+1})} e^{-iM\lambda_1 x'} H_1(\lambda_1 r) dx' / r \right\} e^{i\lambda_1 Mx} \quad (D-88)$$

where $r = \sqrt{(x - x')^2 + Ky^2}$, Since the infinite integral is very slowly convergent, we divide it into two parts and change the contour of integration of the infinite integral. Thus

$$\begin{aligned}\varphi_w &= -\frac{i\lambda_1 \sqrt{K} ye^{iM\lambda_1 x}}{4} \Delta\varphi_t \left\{ \left[\int_x^{\infty} + \int_{x_{i_1+1}}^x \right] e^{-i\omega(x'-x_{i_1+1})} e^{-iM\lambda_1 x'} H_1(\lambda_1 r) dx' / r \right\} \\ &= -\frac{i\lambda_1 \sqrt{K} y}{4} \Delta\varphi_t \left\{ e^{i\omega(x_{i_1+1}-x)} \int_x^{\infty} e^{-i\omega(x'-x)/\beta^2} H_1(\lambda_1 r) dx' / r \right. \\ &\quad \left. + e^{i\omega x_{i_1+1} + iM\lambda_1 x} \int_{x_{i_1+1}}^x e^{-i\omega x'/\beta^2} H_1(\lambda_1 r) dx' / r \right\}\end{aligned}\quad (D-89)$$

Now by translating the range of integration of the first integral we obtain

$$-\lambda_1 Ky \int_x^{\infty} e^{-i\omega(x'-x)/\beta^2} H_1(\lambda_1 r) dx' / r \stackrel{!}{=} \frac{\partial}{\partial y} \int_0^{\infty} e^{-i\omega\xi} H_0(\lambda_1 r) d\xi \quad (D-90)$$

where $r = \sqrt{\xi^2 + Ky^2}$. In reference 1, this integral was made more convergent by changing the contour of integration so that we obtain real negative exponentials. The integral then takes the form

$$\begin{aligned}
\frac{\partial}{\partial y} \int_0^\infty e^{-i\omega\xi} H_0(\lambda_1 r) d\xi = \lambda_1 \sqrt{K} y \left\{ i \int_0^{\pi/2} e^{-\omega \sqrt{K} y \cos \theta / \beta^2} J_1(\lambda_1 \sqrt{K} y \sin \theta) d\theta \right. \\
+ \int_0^{\pi/2} \left[e^{-\omega \sqrt{K} y \cos \theta / \beta^2} Y_1(\sqrt{K} y \lambda_1 \sin \theta) \right. \\
+ \left. \left(\frac{2}{\pi} \right) e^{-\omega \sqrt{K} y \cosh \theta / \beta^2} K_1(\lambda_1 \sqrt{K} y \sinh \theta) \right] d\theta \\
+ \left. \frac{2}{\pi} \int_{\pi/2}^\infty e^{-\omega \sqrt{K} y \cosh \theta / \beta^2} K_1(\lambda_1 \sqrt{K} y \sinh \theta) d\theta \right\} \quad (D-91)
\end{aligned}$$

which is computed numerically for each value of y .

The remaining integral is over a finite range and becomes

$$\int_{x_{i_1+1}}^x e^{i\omega x' / \beta^2} H_1(\lambda_1 r) dx / r = \frac{\lambda_1}{2} \int_{x_{i_1+1}}^x e^{i\omega x' / \beta^2} \ell_1(u) dx' \quad (D-92)$$

where

$$u = \frac{\lambda_1^2}{4} \left[(x - x')^2 + K y^2 \right] \quad (D-93)$$

This integral must also be evaluated numerically.

Finally, combining equations (D-91) and (D-92) into equation (D-89) yields

$$\begin{aligned}
\varphi_w = & -\frac{i\lambda_1 y \Delta\varphi_t}{4} \left\{ i \int_0^{\pi/2} e^{-\omega \sqrt{K} y \cos \theta / \beta^2} J_1(\lambda_1 \sqrt{K} y \sin \theta) d\theta \right. \\
& + \int_0^{\pi/2} \left[e^{-\omega \sqrt{K} y \cos \theta / \beta^2} Y_1(\lambda_1 \sqrt{K} y \sin \theta) \right. \\
& + \left. \left. \left(\frac{2}{\pi} \right) e^{-\omega \sqrt{K} y \cos \theta / \beta^2} K_1(\lambda_1 \sqrt{K} y \sinh \theta) \right] d\theta \right. \\
& + \left. \left(\frac{2}{\pi} \right) \int_{\pi/2}^{\infty} e^{-\omega \sqrt{K} y \cosh \theta / \beta^2} K_1(\lambda_1 \sqrt{K} y \sinh \theta) d\theta \right\} \\
& - \frac{i\lambda_1^2 \sqrt{K} y}{8} \Delta\varphi_t e^{i\omega x_{i_1+1} + iM\lambda_1 x} \int_{x_{i_1+1}}^x e^{-i\omega x' / \beta^2} \ell_1(u) dx'
\end{aligned}$$

or

$$\varphi_w = \Delta\varphi_t G(x, y) \quad (D-94)$$

where

$$\begin{aligned}
G(x, y) = & -\frac{i\lambda_1 y}{4} \left\{ i \int_0^{\pi/2} e^{-\omega \sqrt{K} y \cos \theta / \beta^2} J_1(\lambda_1 \sqrt{K} y \sin \theta) d\theta \right. \\
& + \int_0^{\pi/2} \left[e^{-\omega \sqrt{K} y \cos \theta / \beta^2} Y_1(\lambda_1 \sqrt{K} y \sin \theta) \right. \\
& + \left. \left. \left(\frac{2}{\pi} \right) e^{-\omega \sqrt{K} y \cosh \theta / \beta^2} K_1(\lambda_1 \sqrt{K} y \sinh \theta) \right] d\theta \right. \\
& + \left. \left(\frac{2}{\pi} \right) \int_{\pi/2}^{\infty} e^{-\omega \sqrt{K} y \cosh \theta / \beta^2} K_1(\lambda_1 \sqrt{K} y \sinh \theta) d\theta \right\} \\
& - \frac{i\lambda_1^2 \sqrt{K} y}{8} e^{i\omega x_{i_1+1} + iM\lambda_1 x} \int_{x_{i_1+1}}^x e^{-i\omega x' / \beta^2} \ell_1(u) dx
\end{aligned} \quad (D-95)$$

and

$$u = \frac{\lambda_1^2}{4} [(x - x')^2 + Ky^2] \quad (D-96)$$

Since we have assumed steady symmetric flow, the unsteady perturbation potential satisfies

$$\varphi_{ij_m+1} = -\varphi_{ij_m} \quad \varphi_{ij_m+2} = \varphi_{ij_m-1} \quad (D-97)$$

then on the airfoil

$$\begin{aligned} \Delta\varphi_i &= -2\varphi_{ij_m} + c_{s1}(\varphi_{ij_m-1} - \varphi_{ij_m}) \\ &\quad - c_{s2}(\varphi_{ij_m} - \varphi_{ij_m-1}) - (d_{s1}F_i^U + d_{s2}F_i^L) \\ \Delta\varphi_i &= -(c_{s1} + c_{s2} + 2)\varphi_{ij_m} + (c_{s1} + c_{s2})\varphi_{ij_m-1} - (d_{s1}F_i^U + d_{s2}F_i^L) \end{aligned} \quad (D-98)$$

At the trailing edge the Kutta condition requires

$$\Delta\varphi_{1X} + i\omega\Delta\varphi_1 = 0$$

at $x = x_{i_1}$, from equation (37) of reference 1, we have

$$c_{1i}(\Delta\varphi_{i_1+1} - \Delta\varphi_{i_1}) + d_{1i_1}(\Delta\varphi_{i_1} - \Delta\varphi_{i_1-1}) + i\omega\Delta\varphi_{i_1} = 0 \quad (D-99)$$

Solving for $\Delta\varphi_{i_1+1}$ yields

$$\Delta\varphi_t = \Delta\varphi_{i_1+1} = \Delta\varphi_{i_1} \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1}\right) + (d_{1i_1} / c_{1i_1})\Delta\varphi_{i_1-1} \quad (D-100)$$

Using equation (D-98) to define $\Delta\varphi_{i_1}$ and $\Delta\varphi_{i_1-1}$ yields

$$\begin{aligned} \Delta\varphi_t &= \left[1 - (d_{1i_1} + i\omega) / c_{1i_1}\right] \left[-(c_{s1} + c_{s2} + 2)\varphi_{i_1,j_m} \right. \\ &\quad \left. + (c_{s1} + c_{s2})\varphi_{i_1,j_m-1} - (d_{s1}F_{i_1}^U + d_{s2}F_{i_1}^L) \right] \\ &\quad + (d_{1i_1} / c_{1i_1}) \left[-(c_{s1} + c_{s2} + 2)\varphi_{i_1-1,j_m} \right. \\ &\quad \left. + (c_{s1} + c_{s2})\varphi_{i_1-1,j_m-1} - (d_{s1}F_{i_1-1}^U + d_{s2}F_{i_1-1}^L) \right] \end{aligned}$$

Hence we write

$$\Delta\varphi_t = h_1\varphi_{i_1,j_m} + h_2\varphi_{i_1,j_m-1} + h_3\varphi_{i_1-1,j_m} + h_4\varphi_{i_1-1,j_m-1} + R \quad (D-101)$$

where

$$\begin{aligned}
h_1 &= - \left(c_{s1} + c_{s2} + 2 \right) \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1} \right) \\
h_2 &= \left(c_{s1} + c_{s2} \right) \left(1 - d_{1i} / c_{1i_1} - i\omega / c_{1i_1} \right) \\
h_3 &= - \left(c_{s1} + c_{s2} + 2 \right) \left(d_{1i_1} / c_{1i_1} \right) \\
h_4 &= \left(c_{s1} + c_{s2} \right) \left(d_{1i_1} / c_{1i_1} \right) \\
R &= - \left(1 - d_{1i_1} / c_{1i_1} - i\omega / c_{1i_1} \right) \left(d_{s1} F_{i_1}^U + d_{s2} F_{i_1}^L \right) \\
&\quad - \left(d_{1i} / c_{1i} \right) \left(d_{s1} F_{i-1}^U + d_{s2} F_{i-1}^L \right)
\end{aligned} \tag{D-102}$$

Substituting equation (D-101) into equation (D-94) yields the following expressions for the induced flow from the wake at the point x_i, y_j :

$$\varphi_w = H_{1ij} \varphi_{i_1, j_m} + H_{2ij} \varphi_{i_1, j_m-1} + H_{3ij} \varphi_{i_1, j_m-1} + H_{4ij} \varphi_{i_1-1, j_m-1} \tag{D-103}$$

where

$$H_{1ij} = h_1 G(x_i, y_j) \tag{D-104}$$

and similarly for the other quantities.

D.8 DERIVATION OF THE COEFFICIENTS FOR A MESH WITH TWO AXES OF SYMMETRY

To reduce the number of integrals to be evaluated, we consider two lines of symmetry for the rectangular mesh region. We define

$$\psi_d(x_n - x_i, y_j) = \psi_{dn}(x_i, y_j) = \psi_{dnij}$$

$$\psi_\ell(x_i, y_n - y_j) = \psi_{\ell n}(x_i, y_j) = \psi_{\ell nij}$$

with similar relations for ψ_{un} and ψ_{rn} . For i_{\max} even, we see from figure 7 that the following relations hold for points on the left and right boundaries:

$$\psi_{dnij} = \psi_{di_{\max}-n+1, i_{\max}, j}$$

$$\psi_{\ell n1j} = \psi_{rni_{\max}j}$$

$$\psi_{rn1j} = \psi_{\ell ni_{\max}j}$$

Similarly,

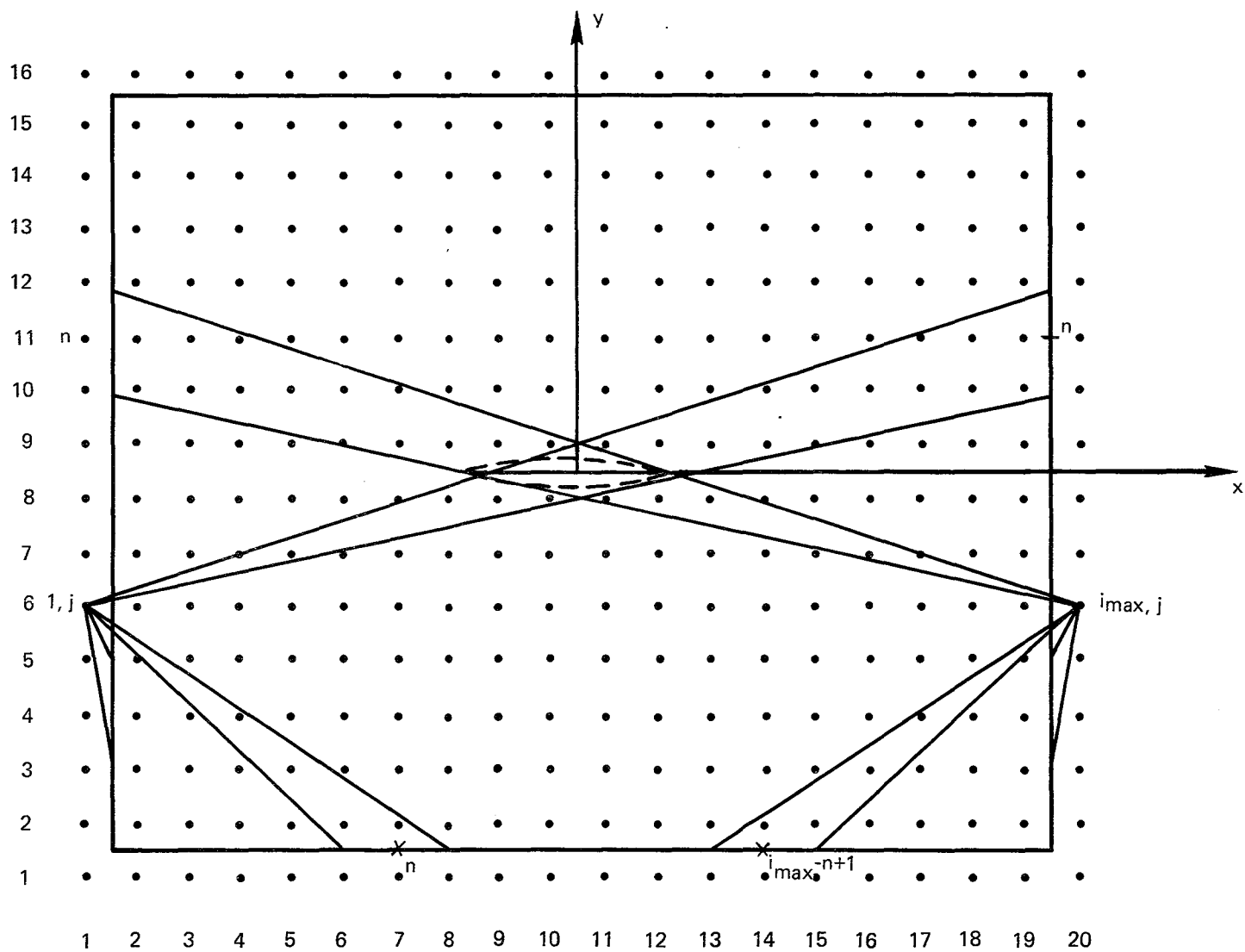


Figure 7.—Illustration of Equal Values of the Integral for Points Associated With the Upstream and Downstream Boundaries for a Grid With Two Axes of Symmetry

$$\psi_{dn2j} = \psi_{di_{\max}^{-n+1}, i_{\max}^{-1}, j}$$

$$\psi_{\ell n2j} = \psi_{rni_{\max}^{-1}, j}$$

$$\psi_{rn2j} = \psi_{\ell ni_{\max}^{-1}, j}$$

On the lower boundary we have from figure 8

$$\psi_{dni1} = \psi_{di_{\max}^{-n+1}, i_{\max}^{-i+1}, 1}$$

$$\psi_{\ell ni1} = \psi_{rni_{\max}^{-i+1}, 1}$$

$$\psi_{rni1} = \psi_{\ell ni_{\max}^{-i+1}, 1}$$

$$\psi_{dni2} = \psi_{di_{\max}^{-n+1}, i_{\max}^{-i+1}, 2}$$

$$\psi_{\ell ni2} = \psi_{rni_{\max}^{-i+1}, 2}$$

$$\psi_{rni2} = \psi_{\ell ni_{\max}^{-i+1}, 2}$$

Because of these equalities we need to compute only the left boundary integrals and the lower boundary integrals. For the left boundary, we have

$$\psi_{dni j} = \psi_{di_{\max}^{-n+1}, i_{\max}^{-i+1}, j} \quad i = 1, 2 \text{ and } j = 2, 3, \dots, j_m \text{ and } n = 2, 3, \dots, i_{\max}^{-1}$$

$$\psi_{\ell ni j} = \psi_{rni_{\max}^{-i+1}, j} \quad \text{for } i = 1, 2 \text{ and } j = 2, 3, \dots, j_m \text{ and } n = 2, 3, \dots, j_m$$

$$\psi_{rni j} = \psi_{\ell ni_{\max}^{-i+1}, j} \quad \text{for } i = 1, 2 \text{ and } j, n = 2, 3, \dots, j_m$$

On the lower boundary we have

$$\psi_{dni j} = \psi_{di_{\max}^{-n+1}, i_{\max}^{-i+1}, j} \quad \text{for } j = 1, 2 \text{ and } i = 2, 3, \dots, i_{\max} / 2 \text{ and } n = 2, 3, \dots, i_{\max}^{-1}$$

$$\psi_{\ell ni j} = \psi_{rni_{\max}^{-i+1}, j}$$

$$\psi_{rni j} = \psi_{\ell ni_{\max}^{-i+1}, j} \quad \text{for } j = 1, 2 \text{ and } i = 2, 3, \dots, i_{\max} / 2 \text{ and for } n = 2, 3, \dots, j_m$$

The total number of ψ integrals to be evaluated are:

1. On left boundary

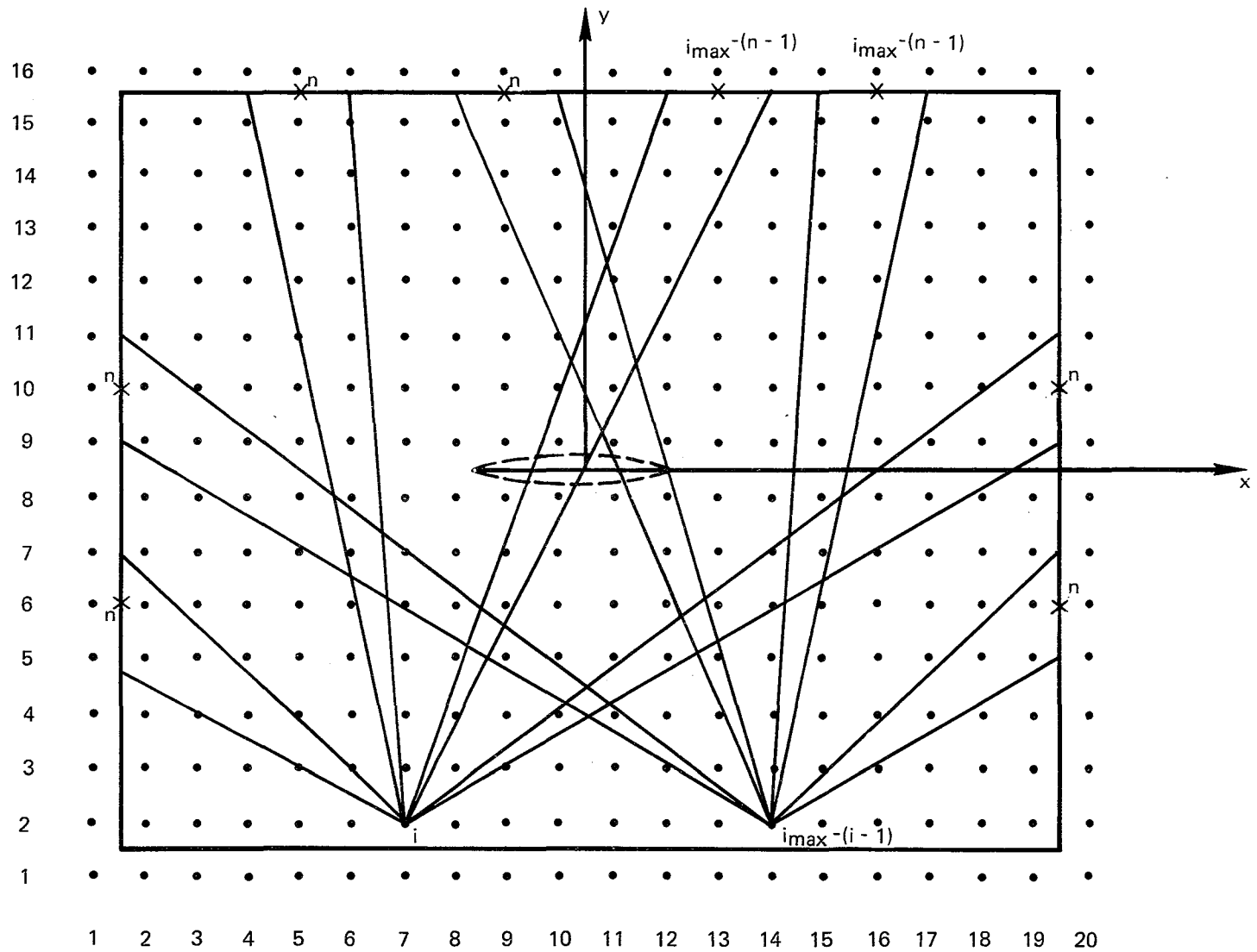


Figure 8.—Illustration of Equal Values of the Integral for Points Associated With the Lower Boundary for a Grid With Two Axes of Symmetry

$$\psi_{dn} : 2(j_m - 1)(i_{\max} - 2)$$

$$\psi_{ln} : 2(j_m - 1)(j_m - 1)$$

$$\psi_{rn} : 2(j_m - 1)(j_m - 1)$$

$$\text{Total} = 2(j_m - 1)[i_{\max} + 2(j_m - 1) - 2]$$

2. On lower boundary

$$\psi_{dn} : 2(i_{\max} / 2 - 1)(i_{\max} - 2) = (i_{\max} - 2)^2$$

$$\psi_{ln} : 2(j_m - 1)(i_{\max} / 2 - 1) = (j_m - 1)(i_{\max} - 2)$$

$$\psi_{rn} : (j_m - 1)(i_{\max} - 2)$$

$$\text{Total on lower boundary} = (i_{\max} - 2)[2(j_m - 1) + i_{\max} - 2]$$

Combining the two totals yields the following total number N_I of integrals to be evaluated:

$$N_I = [i_{\max} + 2(j_m - 1) - 2]^2$$

D.9 DERIVATION OF THE MATRIX ELEMENTS OF THE SYSTEM OF EQUATIONS TO BE SOLVED

For the sake of completeness we write down the difference equations whose coefficients of the φ_{ij} form the elements of part of the matrix. The present program with simpler far-field boundary conditions can be coded to compute these coefficients with small modification.

At elliptic points we have, in the notation of reference 1,

$$a_j \varphi_{ij-1} - (a_j + b_j + E_1 + E_2 - Q_{ij}) \varphi_{ij} + b_j \varphi_{ij+1} + E_1 \varphi_{i+1,j} + E_2 \varphi_{i-1,j} = 0$$

$$\text{for } j = 2, 3, \dots, j_m \text{ and } i = 2, 3, \dots, i_{\max}-1$$

At hyperbolic points,

$$a_j \varphi_{ij-1} - (a_j + b_j - E_3 - Q_{ij}) \varphi_{ij} + b_j \varphi_{ij+1} - (E_3 + E_4) \varphi_{i-1,j} + E_4 \varphi_{i-2,j} = 0$$

$$\text{for } j = 2, 3, \dots, j_m \text{ and } i = 2, 3, \dots, i_{\max}-1$$

For $j = j_m$, we have $\varphi_{ij+1} = -\varphi_{ij}$, and the two equations become

$$a_j \varphi_{ij-1} - (a_j + 2b_j + E_1 + E_2 - Q_{ij}) \varphi_{ij} + E_1 \varphi_{i+1,j} + E_2 \varphi_{i-1,j} = 0$$

$$a_j \varphi_{ij-1} - (a_j + 2b_j - E_3 - Q_{ij}) \varphi_{ij} - (E_3 + E_4) \varphi_{i-1,j} + E_4 \varphi_{i-2,j} = 0$$

For $j = j_m$, we also have boundary conditions for $i_0 \leq i \leq i_1$ and jump conditions for $i > i_1$.

The coding is already written for the difference equations and we need only consider the boundary conditions on the mesh edges. Thus we add to the difference equations the boundary conditions

$$\begin{aligned} \varphi_{1j} = & \sum_{n=2}^{j_m} \left[\alpha_0 \left(\psi_{\ell n 1j}^+ e^{i\lambda_1 M x_1} + \psi_{\ell n 2j}^+ e^{i\lambda_1 M x_2} \right) \right. \\ & \left. - \psi_{\ell n 1j}^- e^{i\lambda_1 M x_1} - \delta_{nj} e^{i\lambda_1 M a_1} (x - x_2) / (4\sqrt{K}) \right] \sigma_{\ell n} \\ & + \sum_{n=2}^{i_{\max}-1} \psi_{dn 1j} e^{i\lambda_1 M x_1} \sigma_{dn} + \sum_{n=2}^{j_m} \psi_{rn 1j} e^{i\lambda_1 M x_1} \sigma_{rn} + \psi_{w 1j} \\ & j = 2, 3, \dots, j_m \end{aligned}$$

$$\begin{aligned} \varphi_{2j} = & \sum_{n=2}^{j_m} \left[\bar{\alpha}_0 \left(\psi_{\ell n 1j}^+ e^{i\lambda_1 M x_1} + \psi_{\ell n 2j}^+ e^{i\lambda_1 M x_2} \right) \right. \\ & \left. - \psi_{\ell n 2j}^- e^{i\lambda_1 M x_2} + \delta_{nj} e^{i\lambda_1 M a_1} (x_1 - x_2) / (4\sqrt{K}) \right] \sigma_{\ell n} \\ & + \sum_{n=2}^{i_{\max}-1} \psi_{dn 2j} e^{i\lambda_1 M x_2} \sigma_{dn} \\ & + \sum_{n=2}^{j_m} \psi_{rn 2j} e^{i\lambda_1 M x_2} \sigma_{rn} + \varphi_{w 2j} \\ & \text{for } j = 2, 3, \dots, j_m \end{aligned}$$

Here the function ψ without the plus or minus subscripts designate the combination $\psi^+ - \psi^-$.

$$\begin{aligned} \varphi_{i1} = & e^{i\lambda_1 M x_i} \left\{ \sum_{n=2}^{j_m} \psi_{\ell n i1} \sigma_{\ell n} + \sum_{n=2}^{j_m} \psi_{rn i1} \sigma_{rn} \right. \\ & + \sum_{n=2}^{i_{\max}-1} \left[\left(\psi_{dni1}^+ + \psi_{dni2}^+ \right) / 2 - \psi_{dni1}^- \right. \\ & \left. \left. - \sqrt{K} \delta_{ni} (y_1 - y_2) / 4 \right] \sigma_{dn} \right\} + \varphi_{wi1} \\ & \text{for } i = 2, 3, \dots, i_{\max}-1 \end{aligned}$$

$$\begin{aligned}
\varphi_{i2} = e & \left\{ \sum_{n=2}^{j_m} \psi_{\ell ni2} \sigma_{\ell n} + \sum_{n=2}^{j_m} \psi_{rni2} \sigma_{rn} \right. \\
& + \sum_{n=2}^{i_{\max}-1} \left[\left(\psi_{dni1}^+ + \psi_{dni2}^+ \right) / 2 - \psi_{dni2}^- \right. \\
& \left. \left. + \sqrt{K} \delta_{ni} (y_1 - y_2) / 4 \right] \sigma_{dn} \right\} + \varphi_{wi2} \\
& \text{for } i = 3, 4, \dots, i_{\max}-2 \\
\varphi_{i_{\max}j} = e & \left\{ \sum_{n=2}^{j_m} \psi_{\ell ni_{\max}j} \sigma_{\ell n} + \sum_{n=2}^{i_{\max}-1} \psi_{dni_{\max}j} \sigma_{dn} \right\} \\
& + \sum_{n=2}^{j_m} \left[\alpha_1 \left(\psi_{rni_{\max}j}^+ e^{i\lambda_1 M x_{i_{\max}}} + \psi_{rni_{\max-1,j}} e^{i\lambda_1 M x_{i_{\max}-1}} \right) \right. \\
& \left. - \psi_{rni_{\max}j}^- e^{i\lambda_1 M x_{i_{\max}}} + \delta_{nj} e^{i\lambda_1 M a_2} \left(x_{i_{\max}} - x_{i_{\max}-1} \right) / (4\sqrt{K}) \right] \sigma_{rn} + \varphi_{wi_{\max}j} \\
& \text{for } j = 2, 3, \dots, j_m \\
\varphi_{i_{\max}-1,j} = e & \left\{ \sum_{n=2}^{j_m} \psi_{\ell n, i_{\max}-1,j} \sigma_{\ell n} + \sum_{n=2}^{i_{\max}-1} \psi_{dni_{\max}-1,j} \sigma_{dn} \right\} \\
& + \sum_{n=2}^{j_m} \left[\bar{\alpha}_1 \left(\psi_{rni_{\max}j}^+ e^{i\lambda_1 M x_{i_{\max}}} + \psi_{rni_{\max-1,j}} e^{i\lambda_1 M x_{i_{\max}-1}} \right) \right. \\
& \left. - \psi_{rni_{\max}-1,j}^- e^{i\lambda_1 M x_{i_{\max}-1}} - \delta_{nj} e^{i\lambda_1 M a_2} \left(x_{i_{\max}} - x_{i_{\max}-1} \right) / (4\sqrt{K}) \right] + \varphi_{wi_{\max}-1,j} \\
& j = 2, 3, \dots, j_m
\end{aligned}$$

Here

$$\alpha_0 = 1 / 2 + iM\lambda_1 (x_1 - x_2) / 4$$

$$\alpha_1 = 1 / 2 + iM\lambda_1 (x_{i_{\max}} - x_{i_{\max}-1}) / 4$$

also.

We notice that there are two expressions for φ_{22} and $\varphi_{i_{\max}-1,2}$. The two equations must be equal; hence we obtain for φ_{22}

$$\begin{aligned}
& \sum_{n=2}^{j_m} \left[\bar{\alpha}_0 \psi_{\ell n 12}^+ e^{i\lambda_1 M x_1} + (\bar{\alpha}_0 - 1) \psi_{\ell n 22}^+ e^{i\lambda_1 M x_2} \right. \\
& \quad \left. + \delta_{n2} e^{i\lambda_1 M a_1} (x_1 - x_2) / (4\sqrt{K}) \right] \sigma_{\ell n} \\
& \quad + e^{i\lambda_1 M x_2} \sum_{n=2}^{i_{\max}-1} \left(\psi_{dn 22}^+ - \psi_{dn 21}^+ \right) / 2 \\
& \quad - \sqrt{K} \delta_{n2} (y_1 - y_2) \Big] \sigma_{dn} = 0
\end{aligned}$$

and for $\varphi_{i_{\max}-1, 2}$

$$\begin{aligned}
& \sum_{n=2}^{j_m} \left\{ \left[\bar{\alpha}_1 \psi_{rn, i_{\max}, 2}^+ e^{i\lambda_1 M x_{i_{\max}}} + (\bar{\alpha}_1 - 1) \psi_{rn, i_{\max}-1, 2}^+ e^{i\lambda_1 M x_{i_{\max}-1}} \right. \right. \\
& \quad \left. \left. - \delta_{n2} e^{i\lambda_1 M a_2} (x_{i_{\max}} - x_{i_{\max}-1}) / (4\sqrt{K}) \right] \sigma_{rn} \right. \\
& \quad \left. + e^{i\lambda_1 M x_{i_{\max}-1}} \sum_{n=2}^{i_{\max}-1} \left[\left(\psi_{dni_{\max}-1, 2}^+ - \psi_{dni_{\max}-1, 1}^+ \right) / 2 \right. \right. \\
& \quad \left. \left. - \delta_{ni_{\max}-1} \sqrt{K} (y_1 - y_2) / 4 \right] \sigma_{dn} = 0
\end{aligned}$$

D.10 THE DEFINITION OF THE BASIC VARIABLES AND THE FORMULATION OF THE MATRIX

We write for the equations

$$\sum_{n=1}^N a_{nm} X_n = R_m, \quad m = 1, 2, \dots, N$$

Let $x_n = \varphi_{ij}$ for $n = (i-1)j_m - 1 + j$ for $i < i_{\max}$ and $n = (i_{\max}-1)j_m - 2 + j$ for $i = i_{\max}$.

Let N_p be the total number of potential variables. Then

$$N_p = i_{\max} j_m - 2$$

The total number of $\sigma_{\ell n}$ variables is $j_m - 1$ (see fig. 8)

The total number of σ_{rn} variables is $j_m - 1$

The total number of σ_{dn} variables is $i_{\max} - 2$

Hence the total number of variables σ_n is

$$2(j_m - 1) + i_{\max} - 2$$

Combining this total with the total number of potential variables, we obtain for the total number of variables N

$$N = i_{\max}j_m + 2(j_m - 1) + i_{\max} - 4$$

To summarize the previous discussion, we identify the variables x_n as

$$X_n = \varphi_{ij} \quad \text{for } n = (i - 1)j_m + j - 1 \quad \text{for } i < i_{\max} \quad \text{and}$$

$$n = (i_{\max} - 1)j_m + j - 2 \quad \text{for } i = i_{\max}$$

$$X_n = \sigma_{\ell k} \quad \text{for } n = N_p + k - 1, \quad k = 2, 3, \dots, j_m$$

$$X_n = \sigma_{rk} \quad \text{for } n = (N_p + j_m - 1) + k - 1, \quad k = 2, 3, \dots, j_m$$

$$X_n = \sigma_{dk} \quad \text{for } n = (N_p + 2j_m - 2) + k - 1, \quad k = 2, 3, \dots, j_m$$

We now consider the numbering of the equations which make up the matrix system. The number of difference equations for the inner solution is

$$N_d = (i_{\max} - 2)(j_m - 1)$$

The number of φ boundary conditions is

$$N_{bc} = 4(j_m - 1) + 2(i_{\max} - 2) - 2$$

The total number of equations is then

$$N_e = i_{\max}j_m + 2(j_m - 1) + i_{\max} - 6$$

We therefore require two more equations to complete the system. These equations are provided by making the relations for φ_{22} and $\varphi_{i_{\max}-1, 2}$ equal.

We now define the equation numbers and the corresponding matrix elements.

1. Equations numbered $m = 1, 2, \dots, N_d$ are the difference equations of the inner solution and

$$N_d = (j_m - 1)(i_{\max} - 2)$$

2. Equations numbered $m = N_d + 1, \dots, N_d + j_m - 1$ are the boundary conditions on

$$\varphi_{1j}, \quad j = 2, 3, \dots, j_m$$

$N_{b1} = N_d + j_m - 1$ is the number of difference equations + the number of φ_{1j} boundary conditions.

3. Equations $N_{b1} + 1$ to $N_{b1} + j_m - 1$ are the boundary conditions on φ_{2j} , $j = 2, 3, \dots, j_m$

$N_{b2} = N_{b1} + j_m - 1$ is the number of difference equations + number of φ_{1j} + number of φ_{2j} boundary conditions

4. Equations $m = N_{b2} + 1$ to $N_{b2} + j_m - 1$ are the boundary conditions on $\varphi_{i_{\max}-1, j}$ boundary conditions.

$N_{b3} = N_{b2} + j_m - 1$ is the number of difference equations + the number of φ_{1j} + the number of φ_{2j} + the number of $\varphi_{i_{\max}-1j}$ boundary conditions.

5. Equations $m = N_{b3} + 1$ to $N_{b3} + j_m - 1$ are boundary conditions on $\varphi_{i_{\max}j}$, $j = 2, 3, \dots, j_m$

$N_{b4} = N_{b3} + j_m - 1$ is the number of difference equations + the number of φ_{1j} + the number of φ_{2j} + the number of $\varphi_{i_{\max} - 1}$ + the number of $\varphi_{i_{\max}j}$ boundary conditions.

6. Equations $m = N_{b4} + 1$ to $N_{b4} + i_{\max} - 2$ are boundary conditions on

$$\varphi_{i1}, \quad i = 2, 3, \dots, i_{\max} - 1$$

$N_{b5} = N_{b4} + j_m - 1$ is the number of difference equations + the number of φ_{i1} + the number of φ_{2j} + the number of $\varphi_{i_{\max}-1j}$ + the number of $\varphi_{i_{\max}j}$ + the number of φ_{i1} boundary conditions

7. Equations $m = N_{b5} + 1$ to $N_{b5} + i_{\max} - 4$ are boundary conditions on

$$\varphi_{i2}, \quad i = 3, 4, \dots, i_{\max} - 2$$

The total number of equations is

$$N = N_{b5} + i_{\max} - 4 = N_{b4} + 2i_{\max} - 6$$

$$N = N_d + 4(j_m - 1) + 2i_{\max} - 6$$

$$= i_{\max}j_m + 2(j_m - 1) + i_{\max} - 6$$

We require two more equations, since the total number of variables is

$i_{\max}j_m + 2(j_m - 1) + i_{\max} - 4$. These equations are obtained from equating the two relations which give φ_{22} and also give $\varphi_{i_{\max}-12}$.

Since the wake integral is involved in all equations greater than $m = N_d$, we need to identify the φ variables associated with it; we have

$$\varphi_{wij} = -\left(H_{1ij}\varphi_{i1j_m} + H_{2ij}\varphi_{i1j_m-1} + H_{3ij}\varphi_{i1-1,j_m} + H_{4ij}\varphi_{i1-1,j_m-1} + R_{ij}\right)$$

$$\text{Let } n_1 = (i_1 - 1)j_m + j_m - 1 = i_1j_m - 1$$

$$\text{and } n_2 = (i_1 - 2)j_m + j_m - 1 = (i_1 - 1)j_m - 1$$

then

$$\varphi_{wij} = -\left(H_{1ij}X_{n_1} + H_{2ij}X_{n_1-1} + H_{3ij}X_{n_2} + H_{4ij}X_{n_2-1}\right)$$

D.11 FORMULAS FOR MATRIX COEFFICIENTS REQUIRED BY OUTER SOLUTION

We shall now write the equations for the various coefficients of the matrix a_{nm} . For the equations $m = 1$ to N_d , the coefficients are for the inner finite difference solutions and are described in reference (1). We now formulate the boundary condition coefficients a_{nm} resulting from the source distribution of the outer solution.

1. Equations $m = N_d + 1$ to $N_d + j_m - 1$; φ_{1j} boundary condition

$$m = N_d + j - 1, \quad \text{for } j = 2, 3, \dots, j_m$$

- a. Coefficients of φ_{1j} $a_{j-1, N_d+j-1} = 1$

$$a_{n_1, m} = -H_{11j} \quad a_{n_1-1, m} = -H_{21j}$$

$$a_{n_2, m} = -H_{31j} \quad a_{n_2-1, m} = -H_{41j}$$

when $n_1 = (i_1 - 2)(j_m - 1) + j_m$ and $n_2 = (i_1 - 1)(j_m - 1)$ are the variables associated with the potentials about the trailing edge.

- b. Coefficients of $\sigma_{\ell k}$ variables, $k = 2, 3, \dots, j_m$

$$a_{N_p+k-1, m} = \alpha_0 \left(\psi_{\ell k 1j}^+ e^{i\lambda_1 M x_1} + \psi_{\ell k 2j}^+ e^{i\lambda_1 M x_2} \right) - \psi_{\ell k 1j}^- e^{i\lambda_1 M x_1} - \delta_{kj} e^{i\lambda_1 M a_1} (x_1 - x_2) / (4\sqrt{K})$$

- c. Coefficients of σ_{rk} variables, $k = 2, 3, \dots, j_m$

$$a_{N+k-1, m} = \psi_{rk 1j} e^{i\lambda_1 M x_1}$$

- d. Coefficients of σ_{dk} variables, $k = 2, 3, \dots, j_m$

$$a_{N_r+k-1, m} = \psi_{dk 1j} e^{i\lambda_1 M x_1}$$

- e. Right-hand side $R_m = R_{1j}$

Here the functions ψ^\pm are defined in equations (D-16) and (D-17) and ψ without the \pm signs in this section is understood to be the combination $\psi^+ - \psi^-$.

2. Equations $m = N_{b1} + i$ to $N_{b1} + j_m - 1$; φ_{2j} boundary conditions

$$m = N_{b1} + j - 1, \quad j = 2, 3, \dots, j_m$$

- a. Coefficients of φ_{ij} variables

$$a_{j_m+j-1,m} = 1$$

$$a_{n_1,j} = -H_{12j} \quad a_{n_1-1,m} = -H_{22j}$$

$$a_{n_2,j} = -H_{32j} \quad a_{n_2-1,m} = -H_{42j}$$

b. Coefficients of $\sigma_{\ell k}$ variables

$$a_{N_p+k-1,m} = \bar{\alpha}_0 \left(\psi_{\ell k 1j} e^{i\lambda_1 M x_1} + \psi_{\ell k 2j} e^{i\lambda_1 M x_2} \right) - \psi_{\ell k 2j} e^{i\lambda_1 M x_2} + \delta_{kj} e^{i\lambda_1 M a_1} (x_1 - x_2) / (4\sqrt{K})$$

for $k = 2, 3, \dots, j_m$

c. Coefficient of σ_{rk} variables

$$a_{N_q+k-1,m} = \psi_{k2j}$$

for $k = 2, 3, \dots, j_m$

d. Coefficients of σ_{dk} variables

$$a_{N_r+k-1,m} = \psi_{dk2j}$$

for $k = 2, 3, \dots, j_m$

e. Right-hand side $R_m = R_{2j}$

3. Equations $m = N_{b2} + 1$ to $N_{b2} + j_m - 1$; $\varphi_{i_{\max}-1,j}$ boundary condition

$$m = N_{b2} + j - 1 \text{ for } j = 2, 3, \dots, j_m$$

a. Coefficients of φ_{ij} variables

$$a_{i_{\max}-2, j_m+j-1,m} = 1$$

$$a_{n_1,m} = -H_{1i_{\max}-1,j} \quad a_{n_1-1,m} = -H_{2i_{\max}-1,j}$$

$$a_{n_2,m} = -H_{3i_{\max}-1,j} \quad a_{n_2-1,m} = -H_{4i_{\max}-1,j}$$

b. Coefficients of $\sigma_{\ell k}$ variables

$$a_{N_p+k-1,m} = \psi_{\ell k i_{\max}-1,j} e^{i\lambda_1 M x_{i_{\max}-1}}$$

for $k = 2, 3, \dots, j_m$

c. Coefficients of σ_{rk} variables

$$a_{N_{\ell}+k-1,m} = \bar{\alpha}_1 \left(\psi_{rki_{\max}j}^+ e^{i\lambda_1 M x_{i_{\max}}} + \psi_{rki_{\max}^{-1}j}^+ e^{i\lambda_1 M x_{i_{\max}^{-1}}} \right) \\ - \psi_{rki_{\max}^{-1}j}^- e^{i\lambda_1 M x_{i_{\max}^{-1}}} + \delta_{ki} e^{i\lambda_1 M a_2 (x_{i_{\max}} - x_{i_{\max}^{-1}})} / 4\sqrt{K}$$

for $k = 2, 3, \dots, j_m$

d. Coefficients of σ_{dk} variables

$$a_{N_r+k-1,m} = \psi_{dki_{\max}^{-1}j} e^{i\lambda_1 M x_{i_{\max}^{-1}}} \quad \text{for } k = 2, 3, \dots, i_{\max} - 1$$

e. Right-hand side $R_m = R_{i_{\max}^{-1},j}$

4. Equations $m = N_{b3} + 1$ to $N_{b3} + j_m - 1$; $\varphi_{i_{\max}j}$ boundary conditions

$$m = N_{b3} + j - 1 \quad \text{for } j = 2, 3, \dots, j_m$$

a. Coefficients of φ_{ij} variables

$$a_{i_{\max}^{-1}j_m+j-2,m} = 1$$

$$a_{n1,m} = -H_{1i_{\max},j} \quad a_{n1-1,m} = -H_{2i_{\max},j}$$

$$a_{n2,m} = -H_{3i_{\max},j} \quad a_{n2-1,m} = -H_{4i_{\max},j}$$

b. Coefficients of $\sigma_{\ell k}$ variables

$$a_{N_p+k-1,m} = e^{i\lambda_1 M x_{i_{\max}}} \psi_{\ell ki_{\max}j}$$

for $k = 2, 3, \dots, j_m$

c. Coefficients of σ_{rk} variables

$$a_{N_{\ell}+k-1,m} = \alpha_1 \left(\psi_{rki_{\max}j}^+ e^{i\lambda_1 M x_{i_{\max}}} + \psi_{rki_{\max}^{-1},j}^+ e^{i\lambda_1 M x_{i_{\max}^{-1}}} \right) \\ - \psi_{rki_{\max}j}^- e^{i\lambda_1 M x_{i_{\max}}} + \delta_{kj} e^{i\lambda_1 M a_2 (x_{i_{\max}} - x_{i_{\max}^{-1}})} / (4\sqrt{K})$$

for $k = 2, 3, \dots, j_m$

d. Coefficients of σ_{dk} variables

$$a_{N_r+k-1,m} = e^{i\lambda_1 M x_{i_{\max}}} \psi_{dki_{\max}j}$$

for $k = 2, 3, \dots, i_{\max} - 1$

c. Right-hand side $R_m = R_{i_{\max}j}$

5. Equations $m = N_{b4} + 1$ to $N_{b4} + i_{\max} - 2 = N_{b5}$

$$m = N_{b4} + i - 1, \quad i = 2, 3, \dots, i_{\max} - 1$$

a. Coefficients of φ_{ij} variable

$$a_{i-1, j_m, m} = 1$$

$$a_{n_1, m} = -H_{1i1} \quad a_{n_1-1, m} = -H_{2i1}$$

$$a_{n_2, m} = -H_{3i1} \quad a_{n_2-1, m} = -H_{4i1}$$

b. Coefficients of $\sigma_{\ell k}$ variables

$$a_{N_p+k-1, m} = e^{i\lambda_1 M x_i} \psi_{\ell ki1}$$

for $k = 2, 3, \dots, j_m$

c. Coefficients of σ_{rk} variables

$$a_{N_\ell+k-1, m} = e^{i\lambda_1 M x_i} \psi_{rki1}$$

$k = 2, 3, \dots, j_m$

d. Coefficients of σ_{dk} variables

$$a_{N_r+k-1, m} = e^{i\lambda_1 M x_i} \left[\left(\psi_{dki1}^+ + \psi_{dki2}^+ \right) / 2 - \psi_{dki1}^- - \sqrt{K} \delta_{ki} (y_1 - y_2) / 4 \right]$$

for $k = 2, 3, \dots, i_{\max} - 1$

e. Right-hand side $R_m = R_{i1}$

6. Equations $m = N_{b5} + 1$ to $N_{b5} + i_{\max} - 4$

$$m = N_{b5} + i - 2, \quad i = 3, \dots, i_{\max} - 2$$

a. Coefficients of φ_{ij} variable

$$a_{i-1, j_m+1, m} = 1$$

$$a_{n_1, m} = -H_{1i2} \quad a_{n_1-1, m} = -H_{2i2}$$

$$a_{n_2, m} = -H_{3i2} \quad a_{n_2-1, m} = -H_{4i2}$$

b. Coefficients of $\sigma_{\ell k}$ variables

$$a_{N_p+k-1,m} = e^{i\lambda_1 M x_i} \psi_{\ell k i 2}$$

for $k = 2, 3, \dots, j_m$

c. Coefficients of σ_{rk} variables

$$a_{N_{\ell}+k-1,m} = e^{i\lambda_1 M x_i} \psi_{r k i 2}$$

$k = 2, 3, \dots, j_m$

d. Coefficients of σ_{dk} variables

$$a_{N_I+k-1,m} = e^{i\lambda_1 M x_i} \left[\left(\psi_{d k i 1}^+ + \psi_{d k i 2}^+ \right) / 2 - \psi_{d k i 2}^- + \sqrt{K} \delta_{ki} (y_1 - y_2) / 4 \right]$$

for $k = 2, 3, \dots, i_{\max} - 1$

e. Right-hand side $R_m = R_{i2}$

7. Equation $m = N_{b5} + i_{\max} - 3$

Matching of two relations giving φ_{22}

a. Coefficients of φ_{ij} variables are zero.

b. Coefficients of $\sigma_{\ell k}$, $k = 2, 3, \dots, j_m$

$$a_{N_p+k-1,m} = \bar{\alpha}_0 \psi_{\ell k 1 2}^+ e^{i\lambda_1 M x_1} + \psi_{\ell k 2, 2}^+ e^{i\lambda_1 M x_2} (\bar{\alpha}_0 - 1) + \delta_{k2} e^{i\lambda_1 M a_1} (x_1 - x_2) / (4\sqrt{K})$$

c. Coefficients of σ_{rk} variables are all zero.

d. Coefficients of σ_{dk} , $k = 2, 3, \dots, i_{\max} - 1$

$$a_{N_I+k-1,m} = e^{i\lambda_1 M x_2} \left[\left(\psi_{d k 2 2}^+ - \psi_{d k 2 1}^+ \right) / 2 - \sqrt{K} \delta_{k2} (y_1 - y_2) / 4 \right]$$

e. Right-hand side $R_m = 0$

8. Equation $m = N_{b5} + i_{\max} - 2$

Matching of two relations giving $\varphi_{i_{\max}-1, 2}$

a. Coefficients of φ_{ij} variables are all zero.

b. Coefficients of $\sigma_{\ell k}$ variables are all zero.

c. Coefficients of σ_{rk} , $k = 2, 3, \dots, j_m$

$$\begin{aligned} a_{N_{\ell}+k-1,m} &= \alpha_1 \psi_{rki_{\max}} 2^e e^{i\lambda_1 M x_{i_{\max}}} \\ &+ (\alpha_1 - 1) \psi_{rki_{\max}-1, 2} e^{i\lambda_1 M x_{i_{\max}-1}} \\ &+ \delta_{k2} e^{i\lambda_1 M a_2} (x_{i_{\max}} - x_{i_{\max}-1}) / (4\sqrt{K}) \end{aligned}$$

d. Coefficients of σ_{dk} , $k = 2, 3, \dots, i_{\max} - 1$

$$\begin{aligned} a_{N_r+k-1,m} &= e^{i\lambda_1 M x_{i_{\max}-1}} \left\{ \left(\psi_{dki_{\max}-1,2} + \psi_{dki_{\max}-1,1} \right) \right. \\ &\quad \left. - \delta_{ki_{\max}-1} \sqrt{K} (y_1 - y_2) / 4 \right\} \end{aligned}$$

e. Right-hand side $R_m = 0$.

The integrals ψ_{dn} , $\psi_{\ell n}$, and ψ_{rn} can be expressed in the form of a single function of several variables resulting in considerable saving in coding. These integrals are represented by

$$\begin{aligned} \psi_{dn}^+(x,y) &= -\frac{1}{4i} \frac{2}{\delta_{n+1} + \delta_n} \left\{ \int_{x_{n-1}}^{x_n} (x' - x_{n-1}) \ell_0(u'_d) dx' / \delta_n \right. \\ &\quad \left. + \int_{x_n}^{x_{n+1}} (x_{n+1} - x') \ell_0(u'_d) dx' / \delta_{n+1} \right\} \end{aligned} \quad (D-105)$$

$$\begin{aligned} \psi_{\ell n}^+(x,y) &= -\frac{1}{4i} \frac{2}{\tilde{\delta}_{n+1} + \tilde{\delta}_n} \left\{ \int_{y_{n-1}}^{y_n} (y' - y_{n-1}) \ell_0(u'_\ell) dy' / \tilde{\delta}_n \right. \\ &\quad \left. + \int_{y_n}^{y_{n+1}} (y_{n+1} - y') \ell_0(u'_\ell) dy' / \tilde{\delta}_{n+1} \right\} \end{aligned} \quad (D-106)$$

$$\begin{aligned} \psi_{rn}^+(x,y) &= -\frac{1}{4i} \frac{2}{\tilde{\delta}_{n+1} + \tilde{\delta}_n} \left\{ \int_{y_{n-1}}^{y_n} (y' - y_{n-1}) \ell_0(u'_r) dy' / \tilde{\delta}_n \right. \\ &\quad \left. + \int_{y_n}^{y_{n+1}} (y_{n+1} - y') \ell_0(u'_r) dy' / \tilde{\delta}_{n+1} \right\} \end{aligned} \quad (D-107)$$

where

$$\delta_n = x_n - x_{n-1}, \quad \tilde{\delta}_n = y_n - y_{n-1}, \quad (D-108)$$

$$u'_d = \lambda_1^2 \left[(x' - x)^2 + K(y + b)^2 \right] / 4 \quad (D-109)$$

$$\begin{aligned} u'_\ell &= \lambda_1^2 \left[(x - a_1)^2 + K(y - y')^2 \right] / 4 \\ &= \lambda_1^2 K \left[(y - y')^2 + (x - a_1)^2 / K \right] / 4 \end{aligned} \quad (D-110)$$

$$\begin{aligned} u'_r &= \lambda_1^2 \left[(x - a_2)^2 + K(y - y')^2 \right] / 4 \\ &= \lambda_1^2 K \left[(y - y')^2 + (x - a_2)^2 / K \right] / 4 \end{aligned} \quad (D-111)$$

and $\ell_0(u) = H_0^{(2)}(\xi)$ with $\xi = 2\sqrt{u}$. The functions $\bar{\psi}_{dn}$, $\bar{\psi}_{\ell n}$, and $\bar{\psi}_{rn}$ can be written down by replacing y by $-y$.

Let

$$u = \mu \left[(x' - \eta)^2 + \xi^2 \right] \quad (D-112)$$

and define the function

$$\begin{aligned} \psi(\mu, \xi, \eta, x_n) &= -\frac{i}{4i} \frac{2}{\delta_{n+1} + \delta_n} \left\{ \int_{x_{n-1}}^{x_n} (x' - x_{n-1}) \ell_0(u) dx' / \delta_n \right. \\ &\quad \left. + \int_{x_n}^{x_{n+1}} (x_{n+1} - x') \ell_0(u) dx' / \delta_{n+1} \right\} \end{aligned} \quad (D-113)$$

where $\delta_n = x_n - x_{n-1}$ as before.

By comparing u in equation (D-112) with u'_d , u'_ℓ , and u'_r in equations (D-109), (D-110), and (D-111) and comparing equations (D-105), (D-106), and (D-109) with equation (D-113), we see that

$$\psi_{dn}^+ = \psi \left(\lambda_1^2 / 4, \sqrt{K}(b + y), x, x_n \right) \quad (D-114)$$

$$\psi_{\ell n}^+ = \psi \left(\lambda_1^2 K / 4, (x - a_1) / \sqrt{K}, y, y_n \right) \quad (D-115)$$

$$\psi_{rn}^+ = \psi \left(\lambda_1^2 K / 4, (\bar{x} - a_2) / \sqrt{K}, y, y_n \right) \quad (D-116)$$

Hence the subscripted quantities become

$$\psi_{\text{dnij}}^+ = \psi \left(\lambda_1^2 / 4, \sqrt{K}(b + y_j), x_i, x_n \right) \quad (\text{D-117})$$

$$\psi_{\text{lnij}}^+ = \psi \left(\lambda_1^2 K / 4, (x_i - a) / \sqrt{K}, y_j, y_n \right) \quad (\text{D-118})$$

$$\psi_{\text{rnij}}^+ = \psi \left(\lambda_1^2 K / 4, (x_i - a_2) / \sqrt{K}, y_j, y_n \right) \quad (\text{D-119})$$

$$\psi_{\text{dnij}}^- = \psi \left(\lambda_1^2 / 4, \sqrt{K}(b - y_j), x_i, x_n \right) \quad (\text{D-120})$$

$$\psi_{\text{lnij}}^- = \psi \left(\lambda_1^2 K / 4, (x_i - a_1) / \sqrt{K}, -y_j, y_n \right) \quad (\text{D-121})$$

$$\psi_{\text{rnij}}^- = \psi \left(\lambda_1^2 K / 4, (x_i - a_2) / \sqrt{K}, -y_j, y_n \right) \quad (\text{D-122})$$

APPENDIX E

AN OBLIQUE COORDINATE SYSTEM FOR SWEPT AND TAPERED WINGS

Consider a vector function \bar{F} with x, y, z , components F_1, F_2, F_3 ; then the divergence of \bar{F} under the transformation of equation (G.1) becomes

$$\nabla \cdot \bar{F} = \frac{\partial \bar{F}}{\partial \xi} \cdot \nabla g + F_{2\eta} + F_{3\eta} \quad (\text{E-1})$$

where $g = (g_x, g_y)$, and F_1, F_2, F_3 are the x, y, z components of \bar{F} . Expressing the operator in conservation form yields

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial \xi} (\bar{F} \cdot \nabla g) + F_{2\eta} + F_{3\eta} - \bar{F} \cdot \frac{\partial}{\partial \xi} (\nabla g) \quad (\text{E-2})$$

Now

$$\nabla g = (1 / c(y), \xi c' / c - x'_{\xi e}(y) / c) \quad (\text{E-3})$$

Here c may be written as $c(\eta)$ and we find that

$$\frac{\partial}{\partial \xi} \nabla g = (0, -c' / c) \quad (\text{E-4})$$

We also have

$$\bar{F} = (u\varphi_{1x} - 2i\omega\varphi_1 / \epsilon, \varphi_{1y}, \varphi_{1z}) \quad (\text{E-5})$$

or substituting the transformation yields

$$\bar{F} = (u\varphi_{1\xi}g_x - 2i\omega\varphi_1 / \epsilon, \varphi_{1\xi}g_y + \varphi_{1\eta}, \varphi_{1\xi}) \quad (\text{E-6})$$

Since the linear transonic small perturbation equation for unsteady flow can be written in the form

$$\nabla \cdot \bar{F} + q\varphi_1 = 0$$

we obtain from equations (E-3) through (E-6)

$$\frac{\partial}{\partial \xi} \left[g_x (u\varphi_{1\xi}g_x - 2i\omega\varphi_1 / \epsilon) + g_y (\varphi_{1\eta} + g_y\varphi_{1\xi}) \right] + \frac{\partial}{\partial \eta} (\varphi_{1\eta} + g_y\varphi_{1\xi}) \quad (\text{E-7})$$

$$+ \varphi_{1\xi\xi} + (c' / c) (\varphi_{1\eta} + g_y\varphi_{1\xi}) + q\varphi_1 = 0$$

The first derivative terms must be changed to reduce the equation to conservation form. Thus

$$\begin{aligned} \frac{c'}{c} \varphi_{1\eta} &= \frac{\partial}{\partial \eta} \left(\frac{c'}{c} \varphi_1 \right) - \left(\frac{c'}{c} \right)' \varphi_1 \\ g_y \varphi_{1\xi} &= \frac{\partial}{\partial \xi} (g_y \varphi_1) + \left(\frac{c'}{c} \right) \varphi_1 \end{aligned}$$

The conservation form of the differential equation then becomes

$$\begin{aligned} & \frac{\partial}{\partial \xi} \left[\left(g_x^2 u + g_y^2 \right) \varphi_{1\xi} + \left(g_y + 2i\omega / \epsilon \right) \varphi_1 + g_y \varphi_{1\eta} \right] \\ & + \frac{\partial}{\partial \eta} \left[\left(1 + c' / c \right) \varphi_{1\eta} + g_y \varphi_{1\xi} \right] + \varphi_{1\xi\xi} \\ & + [q + c' / c - (c' / c)'] \varphi_1 = 0 \end{aligned} \quad (E-8)$$

For the coefficient of φ_1 we see that the second derivative of the chord in the spanwise variable must be continuous for the conservation form to be valid. If the nonconservation form is used, there is not this restriction on the planform and we obtain

$$\begin{aligned} & g_x (\partial / \partial \xi) \left[g_x u \varphi_{1\xi} - 2i\omega \varphi_1 / \epsilon \right] + g_y (\partial / \partial \xi) (\varphi_{1\eta} + g_y \varphi_{1\xi}) \\ & + (\partial / \partial \eta) (\varphi_{1\eta} + g_y \varphi_{1\xi}) + \varphi_{1\xi\xi} + q \varphi_1 = 0 \end{aligned} \quad (E-9)$$

The condition that the equation be hyperbolic for both forms is

$$g_x^2 u + g_y^2 < 0 \quad (E-10)$$

The root chord of the wing must be a plane of symmetry and we must impose the conditions that $\varphi_{1y} = 0$ at $y = \eta = 0$. In terms of the ξ, η variables this becomes

$$\varphi_{1\eta} + g_y \varphi_{1\xi} = 0 \quad \text{for } y = \eta = 0 \quad (E-11)$$

Since this term is zero we must also have, for small η_2 ,

$$(\varphi_{1\eta} + g_y \varphi_{1\xi})_{\eta = -\eta_2/2} \approx - (\varphi_{1\eta} + g_y \varphi_{1\xi})_{\eta = \eta_2/2}$$

and the difference form of the η derivative becomes, at $\eta = 0$,

$$\frac{\partial}{\partial \eta} (\varphi_{1\eta} + g_y \varphi_{1\xi}) = \frac{(\varphi_{1\eta} + g_y \varphi_{1\xi})_{\eta = \eta_2/2}}{\eta_2} \quad (E-12)$$

If we introduce the quantities

$$\mu = g_x = 1 / c(\eta), \quad \nu = g_y = -\xi c'(\eta) / c(\eta) - x'_{le}(\eta) / c(\eta)$$

then the differential equation (eq. E-9) becomes

$$\begin{aligned} & \mu (\partial / \partial \xi) \left[\mu u \varphi_{1\xi} - 2i\omega \varphi_1 / \epsilon \right] + \nu (\partial / \partial \xi) \left[\varphi_{1\eta} + \nu \varphi_{1\xi} \right] \\ & + (\partial / \partial \eta) (\varphi_{1\eta} + \nu \varphi_{1\xi}) + \varphi_{1\xi\xi} + q \varphi_1 = 0 \end{aligned} \quad (E-13)$$

Using the equations (E-11), (E-12), and (E13) yields the following differential equation for points along the root chord $\eta = 0$ of the wing

$$\mu(\partial / \partial \xi) \left[\mu u \varphi_{1\xi} - 2i\omega \varphi_1 / \epsilon \right] + \left[\left(\varphi_{1\eta} + \nu \varphi_{1\xi} \right)_{\eta = \eta_2/2} \right] / \eta_2 + \varphi_{1\xi\xi} + q\varphi_1 = 0 \quad (\text{E-14})$$

Equations (E-13) and (E-14) may be differenced in the same way as the differential equations in reference 2. Thus for the first derivative in ξ , we have

$$\varphi_{1\xi} = c_{1i} (\varphi_{i+1, jk} - \varphi_{ijk}) + d_{1i} (\varphi_{ijk} - \varphi_{i-1, jk}) \quad (\text{E-15})$$

where from equation (H-20) on page 40 of reference 2

$$c_{1i} = (\xi_i - \xi_{i-1}) / [(\xi_{i+1} - \xi_{i-1}) (\xi_{i+1} - \xi_i)] \quad (\text{E-16})$$

$$d_{1i} = (\xi_{i+1} - \xi_i) / [(\xi_{i+1} - \xi_{i-1}) (\xi_i - \xi_{i-1})]$$

Similarly

$$\varphi_{1\eta} = c_{1j} (\varphi_{ij+1, k} - \varphi_{ijk}) + d_{1j} (\varphi_{ijk} - \varphi_{ij-1, k}) \quad (\text{E-17})$$

where c_{1j} and d_{1j} have the same form as equation (E-16) but with η replacing ξ .

Since μ is a function of y and ν is a function of x and y , we may write

$$\mu(\eta_j) = \mu_j \quad \text{and} \quad \nu(\xi_i, \eta_j) = \nu_{ij} \quad \text{at the point } (i, j, k)$$

From equations (19) and (20) of reference 1 we see that

$$\begin{aligned} & \left\{ \mu \frac{\partial}{\partial \xi} \left[\mu u \varphi_{1\xi} - 2i\omega \varphi_1 / \epsilon \right] \right\}_{ijk} \\ &= \mu_j^2 \left[2c_{1i} u_{i+1/2, jk} (\varphi_{i+1jk} - \varphi_{ijk}) \right. \\ & \quad \left. - 2d_{1i} u_{i-1/2, jk} (\varphi_{ijk} - \varphi_{i-1jk}) \right. \\ & \quad \left. - 2\mu_j (i\omega / \epsilon) [c_{1i} (\varphi_{i+1, jk} - \varphi_{ijk}) + d_{1i} (\varphi_{ijk} - \varphi_{i-1, jk})] \right] \end{aligned} \quad (\text{E-18})$$

also

$$\left[\nu \frac{\partial}{\partial \xi} \varphi_{1\eta} \right]_{ijk} = \nu_{ij} [c_{1i} (\varphi_{1\eta_{i+1, jk}} - \varphi_{1\eta_{ijk}}) + d_{1i} (\varphi_{1\eta_{ijk}} - \varphi_{1\eta_{i-1, jk}})] \quad (\text{E-19})$$

Applying the formula in equation (E-15) to each term of equation (E-19) we have, finally,

$$\left[\nu \frac{\partial}{\partial \xi} \varphi_{1\eta} \right]_{ijk} = \nu_{ij} \left\{ c_{1i} \left[c_{1j} (\varphi_{i+1, j+1, k} - \varphi_{i+1, jk}) + d_{1j} (\varphi_{i+1, jk} - \varphi_{i+1, j-1, k}) \right] \right. \\ \left. + (d_{1i} - c_{1i}) \left[c_{1j} (\varphi_{ij+1, k} - \varphi_{ijk}) + d_{1j} (\varphi_{ijk} - \varphi_{ij-1, k}) \right] \right. \\ \left. - d_{1i} \left[c_{1j} (\varphi_{i-1, j+1, k} - \varphi_{i-1, jk}) + d_{1j} (\varphi_{i-1, jk} - \varphi_{i-1, j-1, k}) \right] \right\} \quad (E-20)$$

Similarly, from equations (19) and (20) of page 40 of reference 1 we obtain

$$\nu \frac{\partial}{\partial \xi} (\nu \varphi_{1\xi}) = \nu_{ij} \left[2c_{1i} \nu_{i+1/2, j} (\varphi_{i+1, jk} - \varphi_{ijk}) - 2d_{1i} \nu_{i-1/2, j} (\varphi_{ijk} - \varphi_{i-1, jk}) \right] \quad (E-21)$$

The remaining second derivative terms take the form

$$\varphi_{1\eta\eta} = 2a_j (\varphi_{ij+1, k} - \varphi_{ijk}) - 2b_j (\varphi_{ijk} - \varphi_{ij-1, k}) \quad (E-22)$$

$$\varphi_{1\xi\xi} = 2a_k (\varphi_{ijk+1} - \varphi_{ijk}) - 2b_k (\varphi_{ijk} - \varphi_{ijk-1}) \quad (E-23)$$

The boundary conditions on the wing take the same form as for the cartesian coordinate, since the $\varphi_{1\xi\xi}$ is essentially unchanged from the unswept case derived in reference 1.

Consider the term in equation (E-14)

$$\left[(\varphi_{1\eta} + \nu \varphi_{1\xi})_{\eta = \eta_2/2} \right] / \eta_2 \quad (E-24)$$

We need to express it in terms of the quantities at the grid points. Now

$$\varphi_{1\eta} \big|_{\eta = \eta_2/2} = (\varphi_{i2k} - \varphi_{i1k}) / \eta_2 \quad (E-25)$$

Remembering that $\eta_j = 0$ for $j = 1$ and $\eta = \eta_j$ for $j = 2$, we see that

$$\varphi_{1\eta} \big|_{\eta = \eta_2/2} = c_{1i} (\varphi_{i+1, 2, k} + \varphi_{i+1, 1, k} - \varphi_{i2, k} - \varphi_{i1, k}) \\ + d_{1i} (\varphi_{i2, k} + \varphi_{i1, k} - \varphi_{i-1, 2, k} - \varphi_{i-1, 1, k}) / 2 \quad (E-26)$$

Substituting equations (E-25) and (E-26) into equation (E-24) leads to

$$\left[(\varphi_{1\eta} + \nu \varphi_{1\xi})_{\eta = \eta_2/2} \right] / \eta_2 = (\varphi_{i2, k} - \varphi_{i1, k}) / \eta_2^2 \\ + \nu_{i, 3/2} \left[c_{1i} (\varphi_{i+1, 2, k} - \varphi_{i+1, 1, k} - \varphi_{i2, k} - \varphi_{i1, k}) \right] / 2\eta_2$$

The assumption of plane wave boundary conditions on upstream, downstream, upper and lower boundaries in the cartesian coordinate system yields

$$\varphi_{1_x} + i\omega M \varphi_1 / (1 + M) = 0 \quad \text{at } x = x_{i_{\max}}$$

$$\varphi_{1_x} + i\omega M \varphi_1 / (1 - M) = 0 \quad \text{at } x = x_1$$

$$\varphi_{1_y} - i\lambda_1 \sqrt{K} \varphi_1 = 0 \quad \text{at } y = y_{j_{\max}}$$

$$\varphi_{1_y} + i\lambda_1 \sqrt{K} \varphi_1 = 0 \quad \text{at } y = y_1$$

In the new variables, these relations become

$$\varphi_{1_\xi} + i\omega M \varphi_1 / (1 + M) = 0$$

$$\varphi_{1_\xi} - i\omega M \varphi_1 / (1 - M) = 0$$

$$\varphi_{1_\eta} + \nu \varphi_{1_\xi} + i\lambda_1 \sqrt{K} \varphi_1 = 0$$

$$\varphi_{1_\eta} + \nu \varphi_{1_\xi} - i\lambda_1 \sqrt{K} \varphi_1 = 0$$

From the form of the second derivative terms and particularly the cross-derivative terms we see that the difference equations associated with an interior point involve the 11 points in figure 4 in place of the usual 7 points in figure 5. For the $x = \text{constant}$ line relaxation solution, the matrix is still tridiagonal. For a direct solution, the matrix is still sparse and somewhat banded.

Along the wake, the condition that the vortex sheet not support a load is

$$\Delta \varphi_{1_\xi} + i\omega \Delta \varphi_1 = 0$$

and is thus unchanged from the cartesian form.

APPENDIX F

ROW RELAXATION FOR THREE-DIMENSIONAL FLOW

For hyperbolic points, additional fictitious time-dependent terms must be added to make the row relaxation procedure converge. Under the assumption that the calculations are swept in the direction of increasing j and k , the difference equation (A-2) from reference 2 becomes

$$\begin{aligned}
 & 2E_3 \left(\varphi_{ijk}^{(s)} - \varphi_{i-1, jk}^{(s)} \right) + 2E_4 \left(\varphi_{i-2, jk}^{(s)} - \varphi_{i-1, jk}^{(s)} \right) \\
 & + 2a_{yj} \left(\varphi_{ij-1, k}^{(n)} - \varphi_{ijk}^{(s)} \right) - 2b_{yj} \left(\varphi_{ijk}^{(s)} - \varphi_{ij+1, k}^{(n-1)} \right) \\
 & + 2a_{zk} \left(\varphi_{ijk-1}^{(n)} - \varphi_{ijk}^{(s)} \right) - 2b_{zk} \left(\varphi_{ijk}^{(s)} - \varphi_{ijk+1}^{(n-1)} \right) \\
 & + q_{ijk}^{(s)} \varphi_{ijk}^{(s)} = 0
 \end{aligned} \tag{F-1}$$

where $E_3 = c_{i-1} u_{i-1} / 2jk - i\omega c_{2i} / \epsilon$, $E_4 = d_{i-1} u_{i-3} / 2jk - i\omega d_{2i} / \epsilon$,

and the superscripts n and $n-1$ denote the results of the current relaxation sweep and the previous one, respectively. The superscript s denotes the quantities for which equations (F-1) for all i and for fixed j and k are solved. The subscripted variables a , b and c are defined in reference 2 on page 68.

We now introduce a fictitious time derivative related to the iteration by the relation:

$$\varphi_{ijk}^{(n-1)} = \varphi_{ijk}^{(n)} - \Delta t \left(\varphi_{ijk} \right)_t \tag{F-2}$$

Introducing underrelaxation by a factor r yields

$$\varphi_{ijk}^{(n)} = r \left(\varphi_{ijk}^{(s)} - \varphi_{ijk}^{(n-1)} \right) + \varphi_{ijk}^{(n-1)}$$

Eliminating $\varphi_{ijk}^{(n-1)}$ leads to

$$\varphi_{ijk}^{(s)} = \varphi_{ijk}^{(n)} - \left(\frac{r-1}{r} \right) \Delta t \left(\varphi_{ijk} \right)_t \tag{F-3}$$

by means of equations (F-2) and (F-3), the difference equation can be expressed entirely in terms of the n^{th} iteration for the potential. After dropping the superscript n , we obtain

$$\begin{aligned}
& 2E_3 \left[\varphi_{ijk} - \varphi_{i-1, jk} - \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk} - \varphi_{i-1, jk})_t \right] \\
& + 2E_4 \left[\varphi_{i-2, jk} - \varphi_{i-1, jk} - \left(\frac{r-1}{r} \right) \Delta t (\varphi_{i-2, jk} - \varphi_{i-1, jk})_t \right] \\
& + 2a_{yj} \left[\varphi_{ij-1, k} - \varphi_{ijk} + \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk})_t \right] \\
& - 2b_{yj} \left[\varphi_{ijk} - \left(\frac{r-1}{r} \right) \Delta t \varphi_{ijk}_t - \varphi_{ij+1, k} + \Delta t (\varphi_{ij+1, k})_t \right] \\
& + 2a_{zk} \left[\varphi_{ijk-1} - \varphi_{ijk} + \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk})_t \right] \\
& - 2b_{zk} \left[\varphi_{ijk} - \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk})_t - \varphi_{ijk+1} + \Delta t (\varphi_{ijk+1})_t \right] + q_{ijk} \left[\varphi_{ijk} - \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk})_t \right] = 0
\end{aligned} \tag{F-4}$$

Replacing the difference terms by their appropriate derivatives in preparation for taking the limit as the grid size goes to zero yields

$$- 2b_{yj} \Delta t (\varphi_{ij+1, k} - \varphi_{ijk})_t = - 2b_{yj} (y_{j+1} - y_j) \Delta t \varphi_{yt}$$

Substituting for b_{yj} from equation (A-3) of reference 2 leads to

$$- 2b_{yj} \Delta t (\varphi_{ij+1, k} - \varphi_{ijk})_t = - \frac{2\Delta t}{\Delta y_j} \varphi_{yt} \tag{F-5}$$

where $\Delta y = y_{j+1} - y_{j-1}$. A similar φ_{zt} term results from the b_k term. The φ_{xt} terms resulting from the first two terms of equation (F-4) cancel on taking the limit as Δx , Δy , Δz go to zero. We now proceed to the limit. Neglecting terms of order one and higher in the small increments we obtain the following differential equation from the difference equation (F-4)

$$\begin{aligned}
& \left[u\varphi_{1x} - \frac{2i\omega}{\epsilon} \varphi_1 \right]_x + \varphi_{1yy} + \varphi_{1zz} - 2 \frac{\Delta t}{\Delta y_j} \left(\varphi_{1yt} + \frac{\Delta y_j}{\Delta z_k} \varphi_{1zt} \right) \\
& - 2\Delta t \varphi_{1t} \left[b_{yj} / r - a_{yj}(r-1) / r + b_{zk} / r - a_{zk}(r-1) / r \right] + q\varphi_1 = 0
\end{aligned} \tag{F-6}$$

Let

$$\beta_3 = [b_{yj} + b_{zk} - (a_{yj} + a_{zk})(r-1)] \Delta y_j / r \tag{F-7}$$

then

$$\left[u\varphi_{1x} - 2i(\omega / \epsilon)\varphi_1 \right]_x + \varphi_{1yy} + \varphi_{1zz} - 2 \frac{\Delta t}{\Delta y_j} \left(\varphi_{1yt} + \frac{\Delta y_j}{\Delta z_k} \varphi_{1zt} + \beta_3 \varphi_{1t} \right) + q\varphi_1 = 0 \tag{F-8}$$

The differential equation finally becomes

$$u\varphi_{1_{xx}} + \varphi_{1_{yy}} + \varphi_{1_{zz}} + (u_x - 2i\omega / \epsilon)\varphi_{1_x} - 2 \frac{\Delta t}{\Delta y_j} (\varphi_{1_{yt}} + \beta_{jk} \varphi_{1_{zt}} + \beta_3 \varphi_{1_t}) + q\varphi_1 = 0 \quad (F-9)$$

where

$$\beta_{jk} = \frac{y_{j+1} - y_{j-1}}{z_{k+1} - z_{k-1}}$$

To obtain a convergent operator we add terms $\beta_1 \varphi_{1_{xt}} + \beta_2 \varphi_{1_t}$ to the terms in parentheses and determine β_1 and β_2 to yield a differential equation with the correct time-like behavior for the x coordinate. Let $\tau = \alpha_1 x + \alpha_2 y + \alpha_3 z + t$ then

$$\begin{aligned} \varphi_{1_x} &= \alpha_1 \varphi_{1_\tau} + \varphi_{1_x} & \varphi_{1_{xx}} &= \alpha_1^2 \varphi_{1_{\tau\tau}} + 2\alpha_1 \varphi_{1_{x\tau}} + \varphi_{1_{xx}} \\ \varphi_{1_y} &= \alpha_2 \varphi_{1_\tau} + \varphi_{1_y} & \varphi_{1_{yy}} &= \alpha_2^2 \varphi_{1_{\tau\tau}} + 2\alpha_2 \varphi_{1_{y\tau}} + \varphi_{1_{yy}} \\ \varphi_{1_z} &= \alpha_3 \varphi_{1_\tau} + \varphi_{1_z} & \varphi_{1_{zz}} &= \alpha_3^2 \varphi_{1_{\tau\tau}} + 2\alpha_3 \varphi_{1_{z\tau}} + \varphi_{1_{zz}} \\ \varphi_{1_{xt}} &= \alpha_1 \varphi_{1_{\tau\tau}} + \varphi_{1_{x\tau}} \\ \varphi_{1_{yt}} &= \alpha_2 \varphi_{1_{\tau\tau}} + \varphi_{1_{y\tau}} \\ \varphi_{1_{z\tau}} &= \alpha_3 \varphi_{1_{\tau\tau}} + \varphi_{1_{z\tau}} \end{aligned} \quad (F-10)$$

Substituting equations (F-10) into the differential equation yields

$$\begin{aligned} &u \left(\alpha_1^2 \varphi_{1_{\tau\tau}} + 2\alpha_1 \varphi_{1_{x\tau}} + \varphi_{1_{xx}} \right) + \alpha_2^2 \varphi_{1_{\tau\tau}} + 2\alpha_2 \varphi_{1_{y\tau}} + \varphi_{1_{yy}} \\ &+ \alpha_3^2 \varphi_{1_{\tau\tau}} + 2\alpha_3 \varphi_{1_{z\tau}} + \varphi_{1_{zz}} + (u_x - 2i\omega / \epsilon) (\varphi_{1_x} + \alpha_1 \varphi_{1_\tau}) \\ &- 2 (\Delta t / \Delta y_j) \left[\alpha_2 \varphi_{1_{\tau\tau}} + \varphi_{1_{y\tau}} + \beta_1 (\alpha_1 \varphi_{1_{\tau\tau}} + \varphi_{1_{x\tau}}) + \beta_2 \varphi_{1_\tau} \right. \\ &\left. + \beta_{jk} (\alpha_3 \varphi_{1_{\tau\tau}} + \varphi_{1_{z\tau}}) + \beta_3 \varphi_{1_\tau} \right] + q\varphi_1 = 0 \end{aligned} \quad (F-11)$$

It is necessary to eliminate the cross derivative terms in time τ to reduce the differential equation to canonical form. This requires

$$u\alpha_1 - \beta_1 \Delta t / \Delta y_j = 0$$

$$\alpha_2 - \Delta t / \Delta y_j = 0$$

$$\alpha_3 - \beta_{jk} (\Delta t / \Delta y_j) = 0$$

from which

$$\begin{aligned}\alpha_1 &= \beta_1 (\Delta t / \Delta y_j) / u \\ \alpha_2 &= (\Delta t / \Delta y_j) \\ \alpha_3 &= \beta_{jk} (\Delta t / \Delta y_j)\end{aligned}\tag{F-12}$$

The elimination of the φ_τ terms yields

$$\alpha_1 (u_x - 2i\omega / \epsilon) - 2\beta_2 \Delta t / \Delta y_j - 2\beta_3 \Delta t / \Delta y_j = 0$$

The quantity β_2 is given in terms of β_1 and β_3 by substituting $\beta_1 (\Delta t / \Delta y_j) / u$ for α_1 . Thus

$$\beta_2 = \beta_1 (u_x - 2i\omega / \epsilon) / 2u - \beta_3\tag{F-13}$$

In order for the x variable to be time-like the coefficient of must be positive. This yields the following relation

$$\alpha_1^2 u + \alpha_2^2 + \alpha_3^2 - 2 \frac{\Delta t}{\Delta y_j} [\alpha_2 + \beta_1 \alpha_1 + \beta_{jk} \alpha_3] > 0\tag{F-14}$$

Substituting for the α_j terms yields

$$-\left(\frac{\Delta t}{\Delta y_j}\right)^2 \left[\beta_1^2 / u + 1 + \beta_{jk}^2 \right] > 0\tag{F-15}$$

Since $u < 0$, we have $\beta_1^2 > (-u) (1 + \beta_{jk}^2)$ or

$$\beta_1 = c \sqrt{-u} \sqrt{1 + \beta_{jk}^2}, \quad c > 1\tag{F-16}$$

$$\beta_2 = \beta_1 (u_x - 2i\omega / \epsilon) / 2u - \beta_3\tag{F-17}$$

$$\beta_3 = - (y_{j+1} - y_{j-1}) [b_j + b_k - (a_j + a_k)(r - 1)] / r\tag{F-18}$$

We have now determined the values of β_1 and β_2 required to establish a convergent operator. The differential equation which now must be differenced is

$$\left[u\varphi_{1_x} - 2i(\omega / \epsilon)\varphi_1 \right]_x + \varphi_{1_{yy}} + \varphi_{1_{zz}} - 2 \frac{\Delta t}{\Delta y_j} [\beta_1 \varphi_{1_{xt}} + \beta_2 \varphi_{1_t}] + q\varphi_1 = 0\tag{F-19}$$

Now

$$\varphi_{ijk}^{(n-1)} = \varphi_{ijk}^{(n)} - \Delta t \frac{\partial}{\partial t} \left(\varphi_{ijk}^{(n)} \right) \quad (F-20)$$

$$\varphi_{ijk}^{(s)} = \varphi_{ijk}^{(n)} - \left(\frac{r-1}{r} \right) \Delta t \frac{\partial}{\partial t} \left(\varphi_{ijk}^{(n)} \right)$$

Eliminating $\varphi_{ijk}^{(n)}$ yields

$$\varphi_{ijk}^{(s)} - \varphi_{ijk}^{(n-1)} = \frac{1}{r} \Delta t \frac{\partial}{\partial t} \left(\varphi_{ijk}^{(n)} \right) \quad (F-21)$$

$$\Delta t \frac{\partial \varphi_{ijk}}{\partial t} = r \left(\varphi_{ijk}^{(s)} - \varphi_{ijk}^{(n-1)} \right)$$

The resulting difference equation then becomes (see equation (F-1))

$$\begin{aligned} & 2E_3 \left(\varphi_{ijk}^{(s)} - \varphi_{i-1, jk}^{(s)} \right) + 2E_4 \left(\varphi_{i-2, jk}^{(s)} - \varphi_{i-1, jk}^{(s)} \right) \\ & + 2a_j \left(\varphi_{ij-1, k}^{(n)} - \varphi_{ijk}^{(s)} \right) - 2b_j \left(\varphi_{ijk}^{(s)} - \varphi_{ij+1, k}^{(n-1)} \right) \\ & + 2a_k \left(\varphi_{ijk-1}^{(n)} - \varphi_{ijk}^{(s)} \right) - 2b_k \left(\varphi_{ijk}^{(s)} - \varphi_{ijk+1}^{(n-1)} \right) \\ & - \frac{2}{\Delta y_j} \left[r\beta_1 c_{2i} \left(\varphi_{ijk}^{(s)} - \varphi_{ijk}^{(n-1)} - \varphi_{i-1, jk}^{(s)} + \varphi_{i-1, jk}^{(n-1)} \right) \right. \\ & \quad \left. - r\beta_1 d_{1i-1} \left(\varphi_{i-1, jk}^{(s)} - \varphi_{i-1, jk}^{(n-1)} - \varphi_{i-2, jk}^{(s)} + \varphi_{i-2, jk}^{(n-1)} \right) \right. \\ & \quad \left. + \beta_2 r \left(\varphi_{ijk}^{(s)} - \varphi_{ijk}^{(n-1)} \right) / 2 \right] + q\varphi_{ijk}^{(s)} = 0 \end{aligned} \quad (F-22)$$

We now consider the case in which the z or k variable is swept in the direction of decreasing k. The only terms that change in equation (F-1) are

$$2a_k \left(\varphi_{ijk-1}^{(n-1)} - \varphi_{ijk}^{(s)} \right) - 2b_k \left(\varphi_{ijk}^{(s)} - \varphi_{ijk+1}^{(n)} \right) \quad (F-23)$$

Eliminating $\varphi^{(n-1)}$ by equation (F-2) and $\varphi^{(s)}$ by equation (F-3) yields

$$\begin{aligned} & 2a_k \left[\varphi_{ijk-1} - \Delta t (\varphi_{ijk-1})_t - \varphi_{ijk} + \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk})_t \right] \\ & - 2b_k \left[\varphi_{ijk} - \left(\frac{r-1}{r} \right) \Delta t (\varphi_{ijk})_t - \varphi_{ijk+1} \right] \end{aligned} \quad (F-24)$$

Now

$$2a_k \Delta t [\varphi_{ijk} - \varphi_{ijk-1}]_t = 2a_k (z_k - z_{k-1}) \Delta t \varphi_{zt}$$

From equation (A-3) of reference (2) we have

$$2a_{zk} \Delta t [\varphi_{ijk} - \varphi_{ijk-1}]_t = \frac{2}{z_{k+1} - z_{k-1}} \Delta t \varphi_{zt} = \frac{\Delta t}{\Delta z_k} \varphi_{zt}$$

Taking the limit as $\Delta x, \Delta y, \Delta t$ yields for equation (F-24)

$$\varphi_{zz} + 2b_{zk} \left(\frac{r-1}{r} \right) \Delta t \varphi_t - 2a_{zk} \Delta t \varphi_t / r + \frac{\Delta t}{\Delta z_k} \varphi_{zt} \quad (F-25)$$

In place of equation (F-9) we obtain

$$u\varphi_{1_{xx}} + \varphi_{1_{yy}} + \varphi_{1_{zz}} + (u_x - 2i\omega / \epsilon) \varphi_{1_x} - 2(\Delta t / \Delta y_j) (\varphi_{1_{yt}} - \beta_{jk} \varphi_{1_{zt}} + \beta_3 \varphi_{1_t}) + q\varphi_1 = 0 \quad (F-26)$$

where β_3 is now given by

$$\beta_3 = [a_{zk} + b_{yj} - (a_{yz} + b_{zk})(r-1)] \Delta y_j / r \quad (F-27)$$

instead of equation (F-7). Equation (F-26) differs from equation (F-9) in form only in the sign of the φ_{zt} term. Thus the third line in equation (F-12) becomes

$$\alpha_3 = -\beta_{jk} \frac{\Delta t}{\Delta y_j} / 2 \quad (F-28)$$

Equation (F-13) remains unchanged but the sign of $\alpha_3 \beta_{jk}$ in equation (F-14) is changed.

Substitution of equation (F-28) for α_3 , however, yields equation (F-15) unchanged. Thus the only change in β_1 and β_2 is the definition of β_3 by equation (F-27).

The correction terms for three-dimensional flow do not differ in form from the terms derived in reference 2 for two dimensions. The quantity β_1 differs only by the factor

$$\sqrt{1 + (\Delta y_j / \Delta z_k)^2}$$

from the two-dimensional value. The coefficient of the β_2 term contains the additional term contains the additional term from the z derivative (increasing)

$$[b_{zk} - a_{zk}(r-1)] / r$$

analogous to the two-dimensional relation

$$[b_{yj} - a_{yj}(r-1)] / r$$

(This term is given in error in reference 2 as $-a_{yj}(r-1) / r$.)

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